An Adaptive Observer Design for $n+1$ Coupled Linear Hyperbolic PDEs Based on Swapping

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***Abstract*—In this paper, we use swapping design filters to bring systems of $n+1$ partial differential equations of the hyperbolic type to static form. Standard parameter identification laws can then be applied to estimate unknown parameters in the boundary conditions. Proof of boundedness of the adaptive laws are offered, and the results are demonstrated in simulations.**

I. Introduction

TRANSPORT equations described as first order hyperbolic linear partial differential equations (PDEs) are used to model various complex physical systems. Representative engineering applications such as heat exchangers [?], transmission lines [?], oil wells [?], road traffic [?] and multiphase flow [?], [?], to mention a few, involve convection phenomena with a spatio-temporal dynamics. Due to the wide area of applications, such systems have been subject to extensive research during the last decades. We refer the reader to [?], [?], [?] (and references therein) for significant control related results.

Recently, the backstepping method, well known from nonlinear control theory [?], has been extended to PDEs. The key point of this approach is the introduction of an invertible Volterra transformation that maps the original system of PDEs into a simpler target system whose stability is easier to establish. The invertibility of the transformation, allows to state the equivalence of stability properties for the two systems. The backstepping method was initially developed for parabolic PDEs [?], and has later been adopted to first order hyperbolic systems [?]. In [?], it was extended to two coupled first order systems, with the general $n+1$ case derived in [?]. For such systems, n PDEs travel in one direction, with a single PDE convecting in the opposite direction.

The stabilization result proposed in [?] has been extended even further to general $n+m$ systems in [?], with an arbitrary number of PDEs in each direction and both controllers and observers using boundary sensing only have been developed.

Adaptive control using backstepping was investigated for parabolic PDEs in [?], where a certainty equivalence based backstepping scheme was used. Material concerning adaptive backstepping on hyperbolic PDEs, however, is limited. To the best of our knowledge, one can mainly cite [?], where a hyperbolic partial integro-differential equation was adaptively stabilized using boundary sensing only. Later, an adaptive observer for hyperbolic systems was investigated in [?], where additive disturbance terms in the boundary conditions were estimated. The derived method was applied to a problem from underbalanced drilling in the oil industry, estimating uncertain parameters. Additionally, in [?], backstepping was used in conjunction with sliding mode control to design an adaptive controller estimating and taking into account an uncertain parameter in the boundary condition at the same boundary as actuation. Finally, we mention that the observers designed for the disturbance rejection problem in [?], [?] and [?], and for the leak detection problem in [?] can be interpreted as adaptive observers for hyperbolic PDEs.

Generally, in estimation related problems, for both finite and infinite dimensional systems, the well known K-filters (or Kreisselmeier filters) which were initially suggested by G. Kreisselmeier in [?] are used to derive the state estimate as a static linear function of its parameters. This method was later named swapping design and was more thoroughly investigated in [?] for ODEs and extended to PDEs of parabolic type in [?]. To the best of our knowledge, however, it has not previously been applied to PDEs of hyperbolic type.

In this paper, we investigate the problem of estimating the systems states for $n+1$ linear hyperbolic systems of PDEs with one boundary condition containing unknown multiplicative and additive parameters. Measurements are assumed at the boundaries only, with only a single measurement taken at the boundary that is collocated with the unknown parameter. A similar problem was solved in both [?] and [?] for unknown additive parameters, but known multiplicative parameters. The results were in [?] applied to a linearized version of the Drift Flux Model (DFM) for underbalanced drilling to estimate the pressure inside the oil and gas reservoir. As we assume both the multiplicative and additive parameters to be unknown and particularly for the drilling application, our approach allow for the possibility to adapt the rate of gas and oil extraction (called the production index), and the reservoir pressure emerging from linearizing the DFM.

Also, in the approach of the present paper, the number of required measurements on the boundary that is anti-collocated with the uncertain parameters depends on the number of unknown additive terms in the boundary condition, with all of them known, only a single measurement is required. The penalty for omitting the measurements at the boundary anti-collocated with the uncertain parameters is a stronger require-

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The work of H. Anfinsen was funded by VISTA - a basic research program in collaboration between The Norwegian Academy of Science and Letters, and Statoil.

ment of persistency of excitation for parameter convergence.

This paper is organized as follows: in Section II we present the dynamic model and pose the estimation problem. In Section III, we define a set of K-filters that can be used to express the system states as linear, static combinations of the filters and the unknown parameters and some error terms. The error terms are in Section IV shown to converge exponentially to zero, so that the system states expressed using the K-filters and unknown parameters converge to their true values. This result is formally stated in Theorem 1. From the static parameterization of the system states, standard parameter identification laws can be used to estimate the unknown parameters. We consider the normalized gradient update law in Theorem 2, and prove boundedness of the parameter estimates in the general case, and exponential convergence in the presence of persistent excitation. Simulation examples are shown in Section VII, while conclusions and suggested areas for further work are offered in Section VIII.

II. Problem statement

In this paper, we investigate systems of the form

$$\partial_t u_i(x,t) = -\lambda_i(x)\partial_x u_i(x,t) + \sum_{j=1}^n \sigma_{ij}(x)u_j(x,t) + \omega_i(x)v(x,t) \tag{1}$$

$$\partial_t v(x,t) = \mu(x)\partial_x v(x,t) + \sum_{j=1}^n \theta_j(x)u_j(x,t) \tag{2}$$

with boundary conditions

$$u_i(0,t) = q_i v(0,t) + d_i, \quad i = 1\dots n \tag{3}$$

$$v(1,t) = \sum_{j=1}^n \rho_j u_j(1,t) + U(t) \tag{4}$$

where, for $i, j = 1\dots n$

$$\lambda_i, \mu \in C^1([0,1]) \tag{5}$$

$$\sigma_{ij}, \omega_i, \theta_i \in C^0([0,1]) \tag{6}$$

$$q_i, d_i, \rho_i \in \mathbb{R}, \tag{7}$$

and initial conditions

$$v^0(x),\ u_i^0(x) \in \mathcal{L}^2([0,1]),\ i = 1\dots n. \tag{8}$$

The transport speeds are assumed to satisfy, for all $x \in [0,1]$

$$-\mu(x) < 0 < \lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_n(x). \tag{9}$$

The aim is to estimate the boundary parameters q_i and d_i, $i = 1\dots n$ from boundary sensing only, e.g $v(0,t)$ and optionally $u_i(1,t)$, $i = 1\dots n$. The term $U(t)$ can be considered a control input, although closed loop control is not investigated in this paper. A schematic of the structure of the system is depicted in Figure 1.

III. From dynamic to static parametric form

By introducing a set of filters, we bring the dynamic system to a static form. To ease readability, we define, for $i = 1\dots n$

$$\bar{\rho}_i = \begin{cases} \rho_i & \text{if } u_i(1,t) \text{ is not measured} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where ρ_i are the boundary parameters from (4). For $i = 1\dots n$, we introduce the following input filters

$$\partial_t \eta_i(x,t) = -\lambda_i(x)\partial_x \eta_i(x,t) + \sum_{j=1}^n \sigma_{ij}(x)\eta_j(x,t) + \omega_i(x)\phi(x,t) - k_i(x)(v(0,t) - \phi(0,t)) \tag{11}$$

$$\partial_t \phi(x,t) = \mu(x)\partial_x \phi(x,t) + \sum_{j=1}^n \theta_j(x)\eta_j(x,t) - k_{n+1}(x)(v(0,t) - \phi(0,t)) \tag{12}$$

with boundary conditions

$$\eta_i(0,t) = 0, \tag{13}$$

$$\phi(1,t) = \sum_{j=1}^n (\rho_j - \bar{\rho}_j)u_j(1,t) + \sum_{j=1}^n \bar{\rho}_j \eta_j(1,t) + U(t) \tag{14}$$

These filters model how the control input $U(t)$ and measured $u_i(1,t)$ in (4), affect the system (1)–(4). The injection gains $k_i(x)$, $i = 1\dots n+1$ are yet to be designed.

Next, for $i, j = 1\dots n$, we define the filters

$$\partial_t p_{ij}(x,t) = -\lambda_i(x)\partial_x p_{ij}(x,t) + \sum_{k=1}^n \sigma_{ik}(x)p_{kj}(x,t) + \omega_i(x)r_j(x,t) + k_i(x)r_j(0,t) \tag{15}$$

$$\partial_t r_i(x,t) = \mu(x)\partial_x r_i(x,t) + \sum_{k=1}^n \theta_k(x)p_{ki}(x,t) + k_{n+1}(x)r_i(0,t) \tag{16}$$

$$\partial_t w_{ij}(x,t) = -\lambda_i(x)\partial_x w_{ij}(x,t) + \sum_{k=1}^n \sigma_{ik}(x)w_{kj}(x,t) + \omega_i(x)z_j(x,t) + k_i(x)z_j(0,t) \tag{17}$$

$$\partial_t z_i(x,t) = \mu(x)\partial_x z_i(x,t) + \sum_{k=1}^n \theta_k(x)w_{ki}(x,t) + k_{n+1}(x)z_i(0,t). \tag{18}$$

with boundary conditions

$$p_{ij}(0,t) = \begin{cases} v(0,t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{19}$$

$$r_i(1,t) = \sum_{k=1}^n \bar{\rho}_k p_{ki}(1,t), \tag{20}$$

$$w_{ij}(0,t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{21}$$

$$z_i(1,t) = \sum_{k=1}^n \bar{\rho}_k w_{ki}(1,t). \tag{22}$$

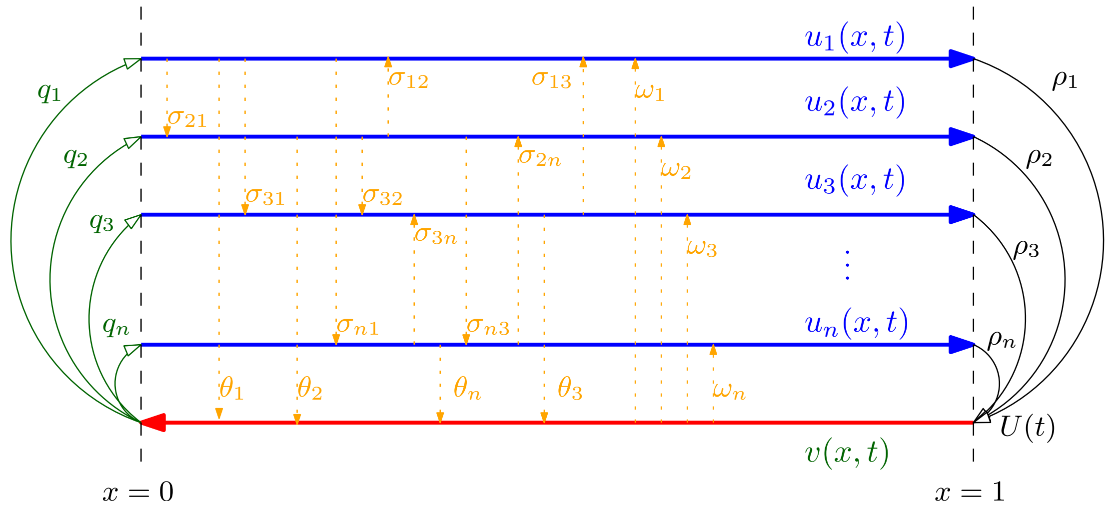

Fig. 1. System structure of u_i (blue) and v (red) with internal couplings (orange), boundary conditions at $x = 0$ (green) and boundary conditions at $x = 1$ (black). The idea for this figure is taken from [**?**].

The filters (15)–(16), (19)–(20) and (17)–(18), (21)–(22) model how the boundary coefficients q_i and additive disturbances d_i, $i = 1 \dots n$ in (3) affect the system (1)–(4), respectively. The filters (11)–(22) are essentially copies of the system dynamics (1)–(2), but with injection terms added. Using the filters, we define the following relations between the filters and the system states. For $i = 1 \dots n$,

$$u_i(x,t) = \bar{u}_i(x,t) + e_i(x,t) \tag{23}$$
$$v(x,t) = \bar{v}(x,t) + \epsilon(x,t) \tag{24}$$

where

$$\bar{u}_i(x,t) = \eta_i(x,t) + \sum_{j=1}^{n} q_j p_{ij}(x,t) + \sum_{j=1}^{n} d_j w_{ij}(x,t) \tag{25}$$
$$\bar{v}(x,t) = \phi(x,t) + \sum_{j=1}^{n} q_j r_j(x,t) + \sum_{j=1}^{n} d_j z_j(x,t) \tag{26}$$

for some error terms $e_i(x,t)$, $i = 1 \dots n$ and $\epsilon(x,t)$. A schematic showing the structure of how the filters relate to the system states is given in Figure 2.

IV. Error dynamics analysis

In this section, we will show that the error terms in (23)–(24) converge to zero exponentially if the injection terms $k_i(x)$, $i = 1 \dots n$ are chosen correctly. The error dynamics can be derived from the static relations (23)–(24) to be

$$\partial_t e_i(x,t) = -\lambda_i(x)\partial_x e_i(x,t) + \sum_{j=1}^{n} \sigma_{ij}(x) e_j(x,t) + \omega_i(x)\epsilon(x,t) + k_i(x)\epsilon(0,t) \tag{27}$$
$$\partial_t \epsilon(x,t) = \mu(x)\partial_x \epsilon(x,t) + \sum_{j=1}^{n} \theta_j(x) e_j(x,t) + k_{n+1}(x)\epsilon(0,t) \tag{28}$$

with boundary conditions

$$e_i(0,t) = 0 \tag{29}$$
$$\epsilon(1,t) = \sum_{j=1}^{n} \bar{\rho}_j e_j(1,t). \tag{30}$$

and initial data

$$e_{i0},\ \epsilon_0 \in \mathcal{L}^2([0,1]),\ i = 1 \dots n. \tag{31}$$

A. Backstepping transformation

The error system dynamics (27)–(28) has the same form as the observer error dynamics in [**?**], where the authors used the following backstepping transformation

$$e_i(x,t) = \alpha_i(x,t) + \int_0^x m_i(x,\xi)\beta(\xi,t)d\xi \tag{32}$$
$$\epsilon(x,t) = \beta(x,t) + \int_0^x m_{n+1}(x,\xi)\beta(\xi,t)d\xi \tag{33}$$

with

$$k_i(x) = -\mu(0) m_i(x,0), \quad i = 1 \dots n+1, \tag{34}$$

to achieve to following target error system

$$\partial_t \alpha_i(x,t) = -\lambda_i(x)\partial_x \alpha_i(x,t) + \sum_{j=1}^{n} \sigma_{i,j}(x)\alpha_j(x,t) + \sum_{j=1}^{n} \int_0^x g_{i,j}(x,\xi)\alpha_j(\xi,t)d\xi \tag{35}$$
$$\partial_t \beta(x,t) = \mu(x)\partial_x \beta(x,t) + \sum_{j=1}^{n} \theta_j(x)\alpha_j(x,t) + \sum_{j=1}^{n} \int_0^x h_j(x,\xi)\alpha_j(\xi,t)d\xi \tag{36}$$

with boundary conditions

$$\alpha_i(0,t) = 0, \text{ for } i = 1, \dots, n \tag{37}$$
$$\beta(1,t) = \sum_{j=1}^{n} \bar{\rho}_j \alpha_j(1,t) \tag{38}$$

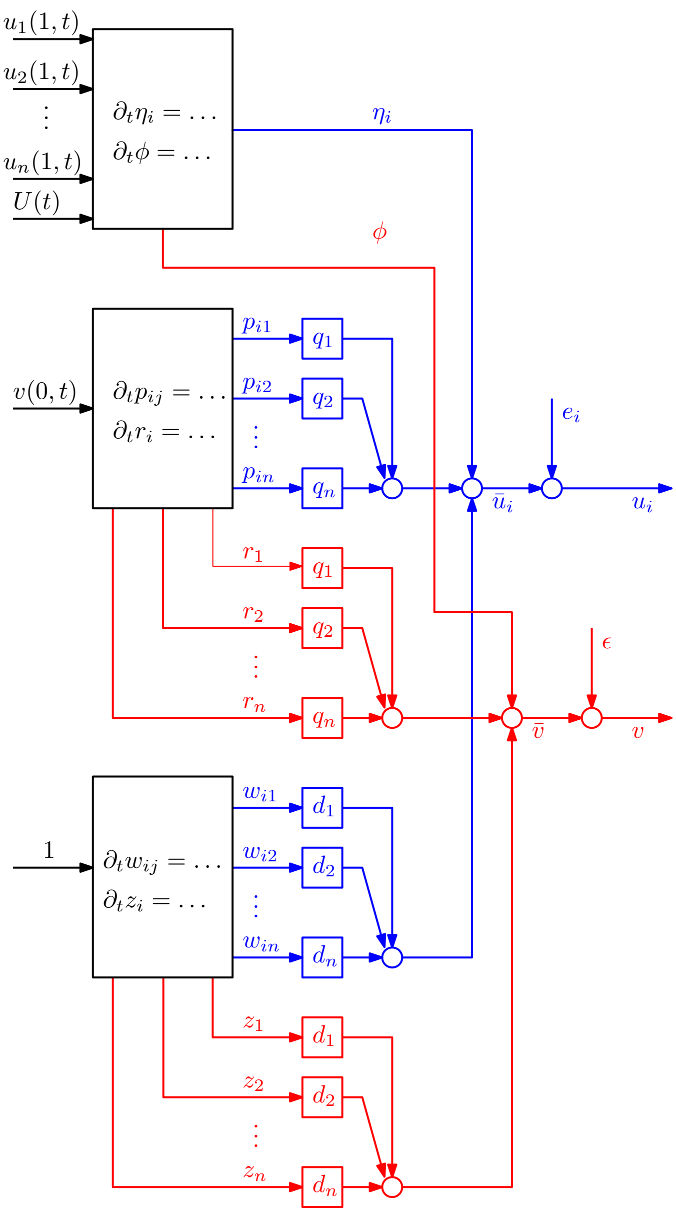

Fig. 2. Structure of the filter system, and their connection to the system states $u_i(x,t)$ (blue) and $v(x,t)$ (red). Only one of the states $u_i(x,t)$ is displayed, and arguments are omitted to ease readability.

where

$$h_i(x,\xi) = -\theta_i(\xi)m_{n+1}(x,\xi) - \int_\xi^x m_{n+1}(x,s)h_i(s,\xi)ds \tag{39}$$

$$g_{i,j}(x,\xi) = -\theta_j(\xi)m_i(x,\xi) - \int_\xi^x m_i(x,s)h_j(s,\xi)ds. \tag{40}$$

With prime denoting the derivative, the kernels in the backstepping transformation (32)–(33) satisfy the following PDEs

$$\lambda_i(x)\partial_x m_i - \mu(\xi)\partial_\xi m_i = \mu'(\xi)m_i + \sum_{j=1}^n \sigma_{i,j}(x)m_j + \omega_i(x)m_{n+1}, \quad i = 1\dots n \tag{41}$$

$$\mu(x)\partial_x m_{n+1} + \mu(\xi)\partial_\xi m_{n+1} = -\mu'(\xi)m_{n+1} + \sum_{i=1}^n \theta_i(x)m_i \tag{42}$$

with boundary conditions

$$m_i(x,x) = \frac{\omega_i(x)}{\lambda_i(x)+\mu(x)}, \quad i = 1\dots n \tag{43}$$

$$m_{n+1}(1,\xi) = \sum_{j=1}^n \bar{\rho}_j m_j(1,\xi) \tag{44}$$

defined over the triangular domain $\mathcal{T} = \{(x,\xi) \mid 0 \le \xi \le x \le 1\}$. It was shown in [?] that the transformation (32)–(33) is invertible, and that the kernel equation (41)–(44) has a unique solution. Since the transformation is invertible, the stability properties of the target and original system are equivalent.

B. Stability

Theorem 1: Under the assumptions

$$\lambda_i, \mu \in C^1([0,1], \mathbb{R}^+), \quad \sigma_{i,j}, \omega_i, \theta_i \in C([0,1]), \tag{45}$$

$$u_i^0, v^0, \eta_i^0, \phi^0, p_{ij}^0, r_i^0, w_{ij}^0, z_i^0 \in \mathcal{L}^2([0,1]) \tag{46}$$

for all $i,j = 1\dots n$, the origin $(e_1, e_2, \dots, \epsilon) = 0$ of the system (27)–(28) together with the boundary conditions (29)–(30) and initial data (31) is exponentially stable in the $\mathcal{L}^2$-sense.

Proof.

We begin by putting the target system (35)–(38) on matrix form as follows

$$\partial_t \alpha(x,t) = -\Lambda(x)\partial_x \alpha(x,t) + \Sigma(x)\alpha(x,t) + \int_0^x G(x,\xi)\alpha(\xi,t)d\xi \tag{47}$$

$$\partial_t \beta(x,t) = \mu(x)\partial_x \beta(x,t) + \theta^T(x)\alpha(x,t) + \int_0^x h^T(x,\xi)\alpha(\xi,t)d\xi \tag{48}$$

with boundary conditions

$$\alpha(0,t) = 0 \tag{49}$$

$$\beta(1,t) = \bar{\rho}^T\alpha(1,t) \tag{50}$$

where

$$\alpha(x,t) = \begin{bmatrix} \alpha_1(x,t) & \dots & \alpha_n(x,t) \end{bmatrix}^T \tag{51}$$

$$\Lambda(x) = \text{diag}\{\lambda_1(x), \dots, \lambda_n(x)\} \tag{52}$$

$$\Sigma(x) = \begin{bmatrix} \sigma_{1,1}(x) & \dots & \sigma_{1,n}(x) \\ \vdots & \ddots & \vdots \\ \sigma_{n,1}(x) & \dots & \sigma_{n,n}(x) \end{bmatrix} \tag{53}$$

$$G(x,\xi) = \begin{bmatrix} g_{1,1}(x,\xi) & \dots & g_{1,n}(x,\xi) \\ \vdots & \ddots & \vdots \\ g_{n,1}(x,\xi) & \dots & g_{n,n}(x,\xi) \end{bmatrix} \tag{54}$$

$$\theta(x) = \begin{bmatrix} \theta_1(x) & \dots & \theta_n(x) \end{bmatrix}^T \tag{55}$$

$$h(x,\xi) = \begin{bmatrix} h_1(x,\xi) & \dots & h_n(x,\xi) \end{bmatrix}^T \tag{56}$$

$$\bar{\rho} = \begin{bmatrix} \bar{\rho}_1 & \dots & \bar{\rho}_n \end{bmatrix}^T. \tag{57}$$

Form the Lyapunov function

$$V_1 = a_1 \int_0^1 e^{-\delta x} \alpha^T(x,t) \Lambda^{-1}(x) \alpha(x,t) dx + \int_0^1 e^{\delta x} \mu^{-1}(x) \beta^2(x,t) dx \tag{58}$$

where a_1 and δ are constants to be decided. For the time derivative of (58), one obtains (see Appendix A1 for details)

$$\dot{V}_1 \leq -(a_1 e^{-\delta} - nM^2) \alpha^T(1,t)\alpha(1,t) - \beta^2(0,t) - \int_0^1 \gamma_1(x) \alpha^T(x,t)\alpha(x,t) dx - \int_0^1 \gamma_2(x) \beta^2(x,t) dx \tag{59}$$

where

$$\gamma_1(x) := a_1 e^{-\delta x} \left(\delta - (2+x) \frac{Mn}{\underline{\lambda}} - \frac{Mn}{\underline{\lambda}\delta} \right) + a_1 \frac{Mn}{\underline{\lambda}\delta} e^{-\delta} - \frac{M}{\underline{\mu}} e^{\delta x} - \frac{M}{\underline{\mu}\delta} e^{\delta} + \frac{M}{\underline{\mu}\delta} e^{\delta x} \tag{60}$$

and

$$\gamma_2(x) := \delta e^{\delta x} - \frac{Mn}{\underline{\mu}} e^{\delta x} - \frac{Mn}{\underline{\mu}} x e^{\delta x} \tag{61}$$

and M, $\underline{\lambda}$, $\underline{\mu}$ are such that, for all $i, j = 1 \dots n$

$$||\sigma_{i,j}||_\infty, ||g_{i,j}||_\infty, ||\theta_i||_\infty, ||h_i||_\infty, ||\rho_i||_\infty < M, \tag{62}$$

$$\lambda_i(x) > \underline{\lambda}, \ \mu(x) > \underline{\mu}, \quad \forall x \in [0,1]. \tag{63}$$

Choosing (see Appendix A2 for details)

$$\delta > \max \left\{ 4\frac{Mn}{\underline{\lambda}}, 2\frac{Mn}{\underline{\mu}}, 1 \right\} \tag{64}$$

and

$$a_1 > \max \left\{ \frac{\left(\frac{M}{\underline{\mu}} e^{\delta} + \frac{M}{\underline{\mu}\delta} e^{\delta} - \frac{M}{\underline{\mu}\delta} \right) e^{\delta}}{\delta - 3\frac{Mn}{\underline{\lambda}} - \frac{Mn}{\underline{\lambda}\delta}}, nM^2 e^{\delta} \right\} \tag{65}$$

concludes the proof. ■

V. UPDATE LAW

From the static form (23)–(24) with error terms converging exponentially to zero, one can use standard gradient and least squares update laws to estimate the unknown parameters q_i and d_i, $i = 1 \dots n$. Suppose $0 \leq m \leq n$ of the $u_i(1,t)$'s are available for measurement and let $k_1 < k_2 < \dots < k_m$ denote their indexes. That is, u_{k_j} for $j = 1, \dots, m$ are measured. Define the following vector of errors

$$e(t) = a(t) - R(t)\nu \tag{66}$$

where

$$R(t) := \begin{bmatrix} P(t) & W(t) \\ r^T(t) & z^T(t) \end{bmatrix} \tag{67}$$

$$P(t) = \begin{bmatrix} p_{k_1,1}(1,t) & \dots & p_{k_1,n}(1,t) \\ p_{k_2,1}(1,t) & \dots & p_{k_2,n}(1,t) \\ \vdots & \ddots & \vdots \\ p_{k_m,1}(1,t) & \dots & p_{k_m,n}(1,t) \end{bmatrix} \tag{68}$$

$$W(t) = \begin{bmatrix} w_{k_1,1}(1,t) & \dots & w_{k_1,n}(1,t) \\ w_{k_2,1}(1,t) & \dots & w_{k_2,n}(1,t) \\ \vdots & \ddots & \vdots \\ w_{k_m,1}(1,t) & \dots & w_{k_m,n}(1,t) \end{bmatrix} \tag{69}$$

$$r(t) = \begin{bmatrix} r_1(0,t) & \dots & r_n(0,t) \end{bmatrix}^T \tag{70}$$

$$z(t) = \begin{bmatrix} z_1(0,t) & \dots & z_n(0,t) \end{bmatrix}^T \tag{71}$$

$$a(t) := \begin{bmatrix} u_{k_1}(1,t) - \eta_{k_1}(1,t) \\ u_{k_2}(1,t) - \eta_{k_2}(1,t) \\ \vdots \\ u_{k_m}(1,t) - \eta_{k_m}(1,t) \\ v(0,t) - \phi(0,t) \end{bmatrix} \tag{72}$$

$$e(t) := \begin{bmatrix} e_{k_1}(1,t) & e_{k_2}(1,t) & \dots & e_{k_m}(1,t) & \epsilon(0,t) \end{bmatrix}^T \tag{73}$$

and where

$$\nu := \begin{bmatrix} q_1 & \dots & q_n & d_1 & \dots & d_n \end{bmatrix}^T \tag{74}$$

is the vector of unknown parameters. Note that all the elements of $a(t)$ and $R(t)$ are either generated using filters or measured.

Theorem 2: Consider the system (1)–(4) with filters (11)–(20) and injection gains given by (34) and (41)–(44). Then the following normalized update law

$$\dot{\hat{\nu}} = \Gamma \frac{R^T(t)(a(t) - R(t)\hat{\nu})}{1 + ||R^T(t)R(t)||^2} \tag{75}$$

for some gain matrix $\Gamma > 0$ ensures that $\tilde{\nu} = \hat{\nu} - \nu \in \mathcal{L}_\infty$. Moreover, if $R(t)$ and $\dot{R}(t)$ are bounded and $R^T(t)$ is persistently exciting (PE), the system parameters converge to their true values exponentially.

Remark 3: A stable plant will ensure that $R(t)$ and $\dot{R}(t)$ are bounded.

Proof of theorem 2. We construct a "prediction error" as follows

$$\hat{e}(t) = a(t) - R(t)\hat{\nu}. \tag{76}$$

Now consider the Lyapunov function candidate

$$V = a_2 V_1 + \frac{1}{2} \tilde{\nu}^T \Gamma^{-1} \tilde{\nu} \tag{77}$$

where $\tilde{\nu} := \hat{\nu} - \nu$ and a_2 is yet to be decided. Then

$$\begin{aligned} \dot{V} &= a_2 \dot{V}_1 + \tilde{\nu}^T \Gamma^{-1} \dot{\hat{\nu}} \\ &\leq -a_2 a_1 e^{-\delta} \alpha^T(1,t)\alpha(1,t) - a_2 \beta^2(0,t) \\ &- a_2 k_1 \int_0^1 \alpha^T(x,t)\alpha(x,t) dx - a_2 k_2 \int_0^1 \beta^2(x,t) dx \\ &+ \tilde{\nu}^T \frac{R^T(t)\hat{e}(t)}{1 + ||R^T(t)R(t)||^2} \end{aligned} \tag{78}$$

where k_1 and k_2 are lower bounds for $\gamma_1(x)$ and $\gamma_2(x)$, respectively. Noticing that $\hat{e}(t) = a(t) - R(t)\hat{\nu} = e(t) - R(t)\tilde{\nu}$, we find

$$\dot{V} \leq -a_2(a_1 e^{-\delta} - nM^2) \alpha^T(1,t)\alpha(1,t) - a_2 \beta^2(0,t) - a_2 k_1 \int_0^1 \alpha^T(x,t)\alpha(x,t) dx - a_2 k_2 \int_0^1 \beta^2(x,t) dx$$

$$
\begin{aligned}
&+ \frac{\tilde{\nu}^T R^T(t) e(t)}{1+||R^T(t)R(t)||^2} - \frac{|R(t)\tilde{\nu}|^2}{1+||R^T(t)R(t)||^2} \\
&\leq -a_2(a_1 e^{-\delta} - nM^2)\alpha^T(1,t)\alpha(1,t) - a_2\beta^2(0,t) \\
&- a_2 k_1 \int_0^1 \alpha^T(x,t)\alpha(x,t)dx - a_2 k_2 \int_0^1 \beta^2(x,t)dx \\
&- \frac{1}{2}\frac{|R(t)\tilde{\nu}|^2}{1+||R^T(t)R(t)||^2} \\
&+ \frac{1}{2}e^T(t)e(t). \qquad (79)
\end{aligned}
$$

We investigate the latter term. We have that

$$
\begin{aligned}
e_i^2(1,t) &= \left(\alpha_i(1,t) + \int_0^1 m^i(1,\xi)\beta(\xi,t)d\xi\right)^2 \\
&\leq 2\alpha_i^2(1,t) + 2\left(\int_0^1 m^i(1,\xi)\beta(\xi,t)d\xi\right)^2 \\
&\leq 2\alpha_i^2(1,t) + 2M^2\left(\int_0^1 \beta(\xi,t)d\xi\right)^2 \\
&\leq 2\alpha_i^2(1,t) + 2M^2\int_0^1 \beta^2(\xi,t)d\xi \qquad (80)
\end{aligned}
$$

where M bounds the kernel $m^i(x,\xi)$. Furthermore

$$\epsilon^2(0,t) = \beta^2(0,t), \qquad (81)$$

thus

$$
\begin{aligned}
\dot{V} &\leq -(a_2(a_1 e^{-\delta} - nM^2) - 1)\alpha^T(1,t)\alpha(1,t) \\
&- \left(a_2 - \frac{1}{2}\right)\beta^2(0,t) \\
&- a_2 k_1 \int_0^1 \alpha^T(x,t)\alpha(x,t)dx \\
&- \int_0^1 (a_2 k_2 - nM^2)\beta^2(x,t)dx \\
&- \frac{1}{2}\frac{|R(t)\tilde{\nu}|^2}{1+||R^T(t)R(t)||^2} \qquad (82)
\end{aligned}
$$

Choose

$$a_2 > \max\left\{\frac{1}{a_1 e^{-\delta} - nM^2}, \frac{nM^2}{k_2}, \frac{1}{2}\right\} \qquad (83)$$

and let δ and a_1 satisfy (64) and (65), respectively. Therefore, V is bounded which in turns implies that $\tilde{\nu}$ is bounded.

Since, from Theorem 1, we have that e_i and ϵ will go to zero exponentially and independently of the update law of Theorem 2, the static form of the measurements in (66) will asymptotically reach the form $a(t) = R(t)\nu$. The latter part of the theorem then immediately follows from part iii) of Theorem 4.3.2 in [**?**]. ∎

VI. Convergence analysis

Having established boundedness of the update law, we will elaborate on some of the requirements for convergence of the estimated parameters. We first claim that if $n_d \leq n$ of parameters d_i are unknown, then a necessary condition for $R^T(t)$ to be PE is that a number $m \geq n_d - 1$ of the $u_i(1,t)$'s are measured, and the unknown parameters must be identifiable through the set of measurements.

This claim can be seen from the fact that the filters (17)–(18) constitute an independent subsystem that is, through (21), driven by constants. Eventually, the filters will reach steady state, and the regressors for the additive terms d_i in (67) will become constants. A total of n_d measurements is therefore required to determine all the additive terms d_i. As $v(0,t)$ must be measured, a total of $m = n_d - 1$ of the parameters $u_i(1,t)$ must also be measured. Lack of identifiability for a parameter will yield a corresponding regressor that is constantly zero.

Following this observation, if all but one d_i are known, only the single measurement $v(0,t)$ is, in theory, required to estimate all the parameters q_i, provided all the requirements of Theorem 2 are met.

VII. Simulations

The system with the adaptive observer were implemented in Matlab for $n = 2$. The kernel equation (41)–(44) was solved using FreeFEM++ [**?**]. Both a stable and an unstable plant were implemented. The stable plant is also simulated for various number of measured $u_i(1,t)$.

A. Stable plant, $u_1(1,t)$ and $u_2(1,t)$ are measured

We here assume that both $u_1(1,t)$ and $u_2(1,t)$ are available for measurement. The system transport speeds were here set to [1]

$$\lambda_1 = \lambda_2 = \mu = 1 \qquad (84)$$

with the in-domain parameters set to

$$\begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \omega_1 \\ \sigma_{2,1} & \sigma_{2,2} & \omega_2 \\ \theta_1 & \theta_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.2 & 0.4 \\ 0 & 0 & 0.2 \\ 0 & 0.2 & 0 \end{bmatrix} \qquad (85)$$

and the boundary parameters set to

$$\begin{bmatrix} q_1 & d_1 & \rho_1 \\ q_2 & d_2 & \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 \\ 1.2 & -0.3 & -0.8 \end{bmatrix}. \qquad (86)$$

The plant's and filters initial values were all set to zero. The following input[2]

$$U(t) = \sin(t) + \sin(\sqrt{2}t) + \sin(\sqrt{3}t) \qquad (87)$$

turned out to make the regressor $R(t)$ satisfy the PE requirements. The adaptation gain was set to

$$\Gamma = 20 \cdot I_{4\times 4}. \qquad (88)$$

The simulation results are found in Figure 3. The estimated parameters reach their true values after approximately 10 seconds of simulation, while the state estimation errors are seen to converge to zero in approximately the same amount of time. The states are also observed to be bounded, which, as stated in Remark 3, ensures that $R(t)$ and $\dot{R}(t)$ are bounded as required by Theorem 2.

B. Stable plant, u_2 unmeasured

We now assume only $u_1(1,t)$ is available for measurement. All the system parameters are as in the previous section. The estimated parameters are shown in Figure 4. As seen from

[1]The plant parameters are the same that were used in the simulation example in [**?**], with the in-domain parameters downscaled to ensure the plant is stable.

[2]The even simpler input $U(t) = \sin(t)$ is also sufficient.

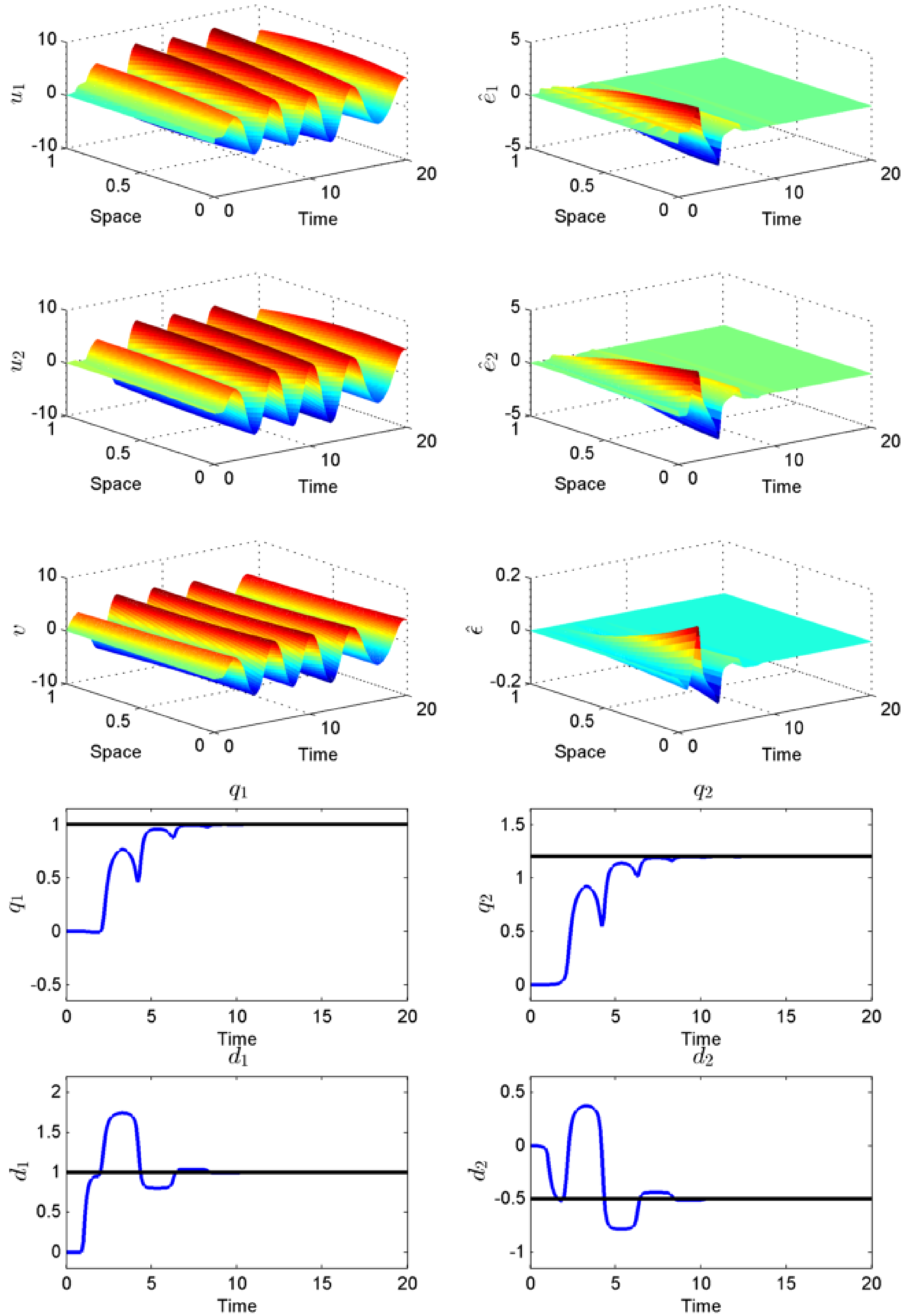

Fig. 3. Estimated states and parameters, $m = 2$

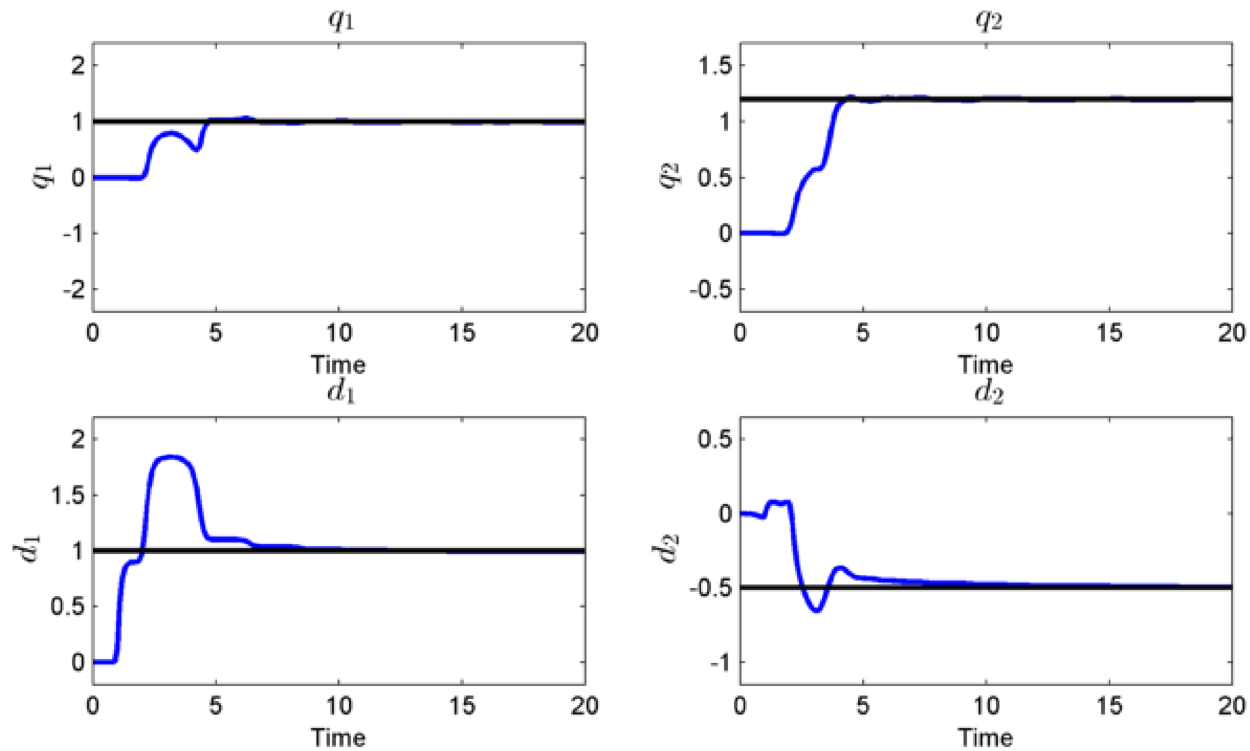

Fig. 4. Estimated parameters, $m = 1$

the figure, all four parameters still converge although the rate of convergence is somewhat slower than when both $u_1(1,t)$ and $u_2(1,t)$ are available for measurement. This observation is most obvious from the estimate of d_2, and is a result of the number of measurements being reduced by one.

C. Stable plant, u_1 unmeasured

Finally, we assume only $u_2(1,t)$ is available for measurement. All other parameters are as in the previous two sections. The estimated parameters are shown in Figure 5. Although the estimates for q_2 and d_2 converge within approximately 8 seconds, the estimates for q_1 and d_1 doesn't even move. This follows from the lack of observability of these to parameters through $u_2(1,t)$ and $v(0,t)$.

D. Unstable plant

The same transport speeds as in the previous cases were used, but the in-domain system parameters were set to

$$\begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \omega_1 \\ \sigma_{2,1} & \sigma_{2,2} & \omega_2 \\ \theta_1 & \theta_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \tag{89}$$

and the boundary parameters were set to

$$\begin{bmatrix} q_1 & d_1 & \rho_1 \\ q_2 & d_2 & \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1.2 & 0 & -0.8 \end{bmatrix}. \tag{90}$$

All other parameters and all initial conditions are the same as in the previous simulation cases. These system parameters yield the same system that was used for demonstrating the theory derived in [**?**], and is open-loop unstable. This is clearly seen from the states displayed in Figure 6. Also observed from the figure, is that the estimates of the multiplicative constants q_1 and q_2 converge. However, the estimates for the additive constants d_1 and d_2 stagnate off their true values. Further investigation reveals that the estimates for d_1 and d_2 stagnates due to the normalization term in (75) ending up very large in magnitude leaving the time derivatives in (75) essentially zero. This follows from the fact that the plant is unstable, which is evident from the system states. An interesting observation from Figure 6, however, is that although the state prediction errors are non-zero, their relative errors compared to the magnitudes of the system states are very small. The relative error for $\hat{e}_1$ for instance, is in the order of 10^{-11}.

VIII. Conclusion

We have developed an adaptive observer for coupled, hyperbolic first-order PDEs with unknown boundary parameters using swapping design. The observer uses a set of filters to convert the dynamic equations into static ones. A straight forward gradient or least squares adaptive law can then be used to estimate the unknown parameters. Proof of boundedness are given, and sufficient conditions ensuring exponential parameter convergence is given. Simulations show parameter convergence for the given conditions.

Areas that could be subject to further work include deriving more explicit conditions on $v(0,t)$ or $U(t)$ to ensure PE and parameter convergence. The observer can also be combined with a controller to establish closed loop adaptive control.

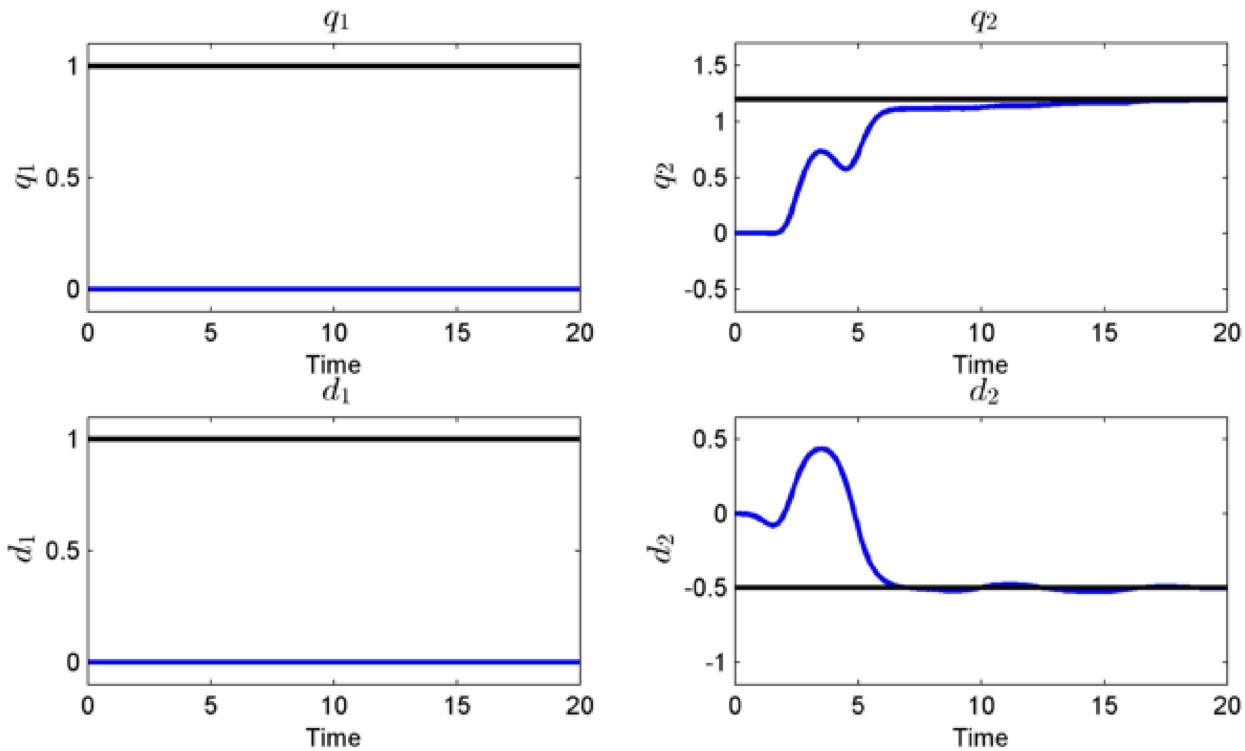

Fig. 5. Estimated parameters, $m = 0$

IX. ACKNOWLEDGMENTS

The authors would like to thank Dr. Ababacar Diagne at Uppsala University for help on numerically solving the kernel equations.

APPENDIX

A. Details regarding the proof of Theorem 1

1) Details regarding Equation (59)*:* Differentiating (58) with respect to time and inserting the dynamics of the target system (35)–(36), we find

$$\begin{aligned}\dot{V}_1 = &-2\int_0^1 a_1 e^{-\delta x}\alpha^T(x,t)\partial_x\alpha(x,t)dx \\ &+2\int_0^1 a_1 e^{-\delta x}\alpha^T(x,t)\Lambda^{-1}(x)\Sigma(x)\alpha(x,t)dx \\ &+2\int_0^1 a_1 e^{-\delta x}\alpha^T(x,t)\Lambda^{-1}(x)\int_0^x G(x,\xi)\alpha(\xi,t)d\xi dx \\ &+2\int_0^1 e^{\delta x}\beta(x,t)\partial_x\beta(x,t)dx \\ &+2\int_0^1 e^{\delta x}\mu^{-1}(x)\beta(x,t)\theta^T(x)\alpha(x,t)dx \\ &+2\int_0^1 e^{\delta x}\mu^{-1}(x)\beta(x,t)\int_0^x h^T(x,\xi)\alpha(\xi,t)d\xi dx \end{aligned} \tag{91}$$

We find the following bounds on the different terms.

a) Part 1:

$$\begin{aligned}&-2a_1\int_0^1 e^{-\delta x}\alpha^T(x,t)\partial_x\alpha(x,t)dx \\ &= -a_1 e^{-\delta}\alpha^T(1,t)\alpha(1,t) \\ &\quad - a_1\delta\int_0^1 e^{-\delta x}\alpha^T(x,t)\alpha(x,t)dx\end{aligned} \tag{92}$$

b) Part 2:

$$\begin{aligned}&2a_1\int_0^1 e^{-\delta x}\alpha^T(x,t)\Lambda^{-1}(x)\Sigma(x)\alpha(x,t)dx \\ &\leq 2a_1\frac{Mn}{\bar{\lambda}}\int_0^1 e^{-\delta x}\alpha^T(x,t)\alpha(x,t)dx\end{aligned} \tag{93}$$

c) Part 3:

$$\begin{aligned}&2a_1\int_0^1 e^{-\delta x}\alpha^T(x,t)\Lambda^{-1}(x)\int_0^x G(x,\xi)\alpha(\xi,t)d\xi dx \\ &\leq a_1\frac{Mn}{\bar{\lambda}}\int_0^1 x e^{-\delta x}\alpha^T(x,t)\alpha(x,t)dx \\ &+ a_1\frac{Mn}{\bar{\lambda}\delta}\int_0^1 (e^{-\delta x}-e^{-\delta})\alpha^T(x,t)\alpha(x,t)dx\end{aligned} \tag{94}$$

d) Part 4:

$$\begin{aligned}&2\int_0^1 e^{\delta x}\beta(x,t)\partial_x\beta(x,t)dx \\ &\leq -\beta^2(0,t) + M^2 n\alpha(1,t)^T\alpha(1,t) \\ &\quad - \delta\int_0^1 e^{\delta x}\beta^2(x,t)dx\end{aligned} \tag{95}$$

e) Part 5:

$$\begin{aligned}&2\int_0^1 e^{\delta x}\mu^{-1}(x)\beta(x,t)\theta^T(x)\alpha(x,t)dx \\ &\leq \frac{Mn}{\bar{\mu}}\int_0^1 e^{\delta x}\beta^2(x,t)dx \\ &\quad + \frac{M}{\bar{\mu}}\int_0^1 e^{\delta x}\alpha^T(x,t)\alpha(x,t)dx\end{aligned} \tag{96}$$

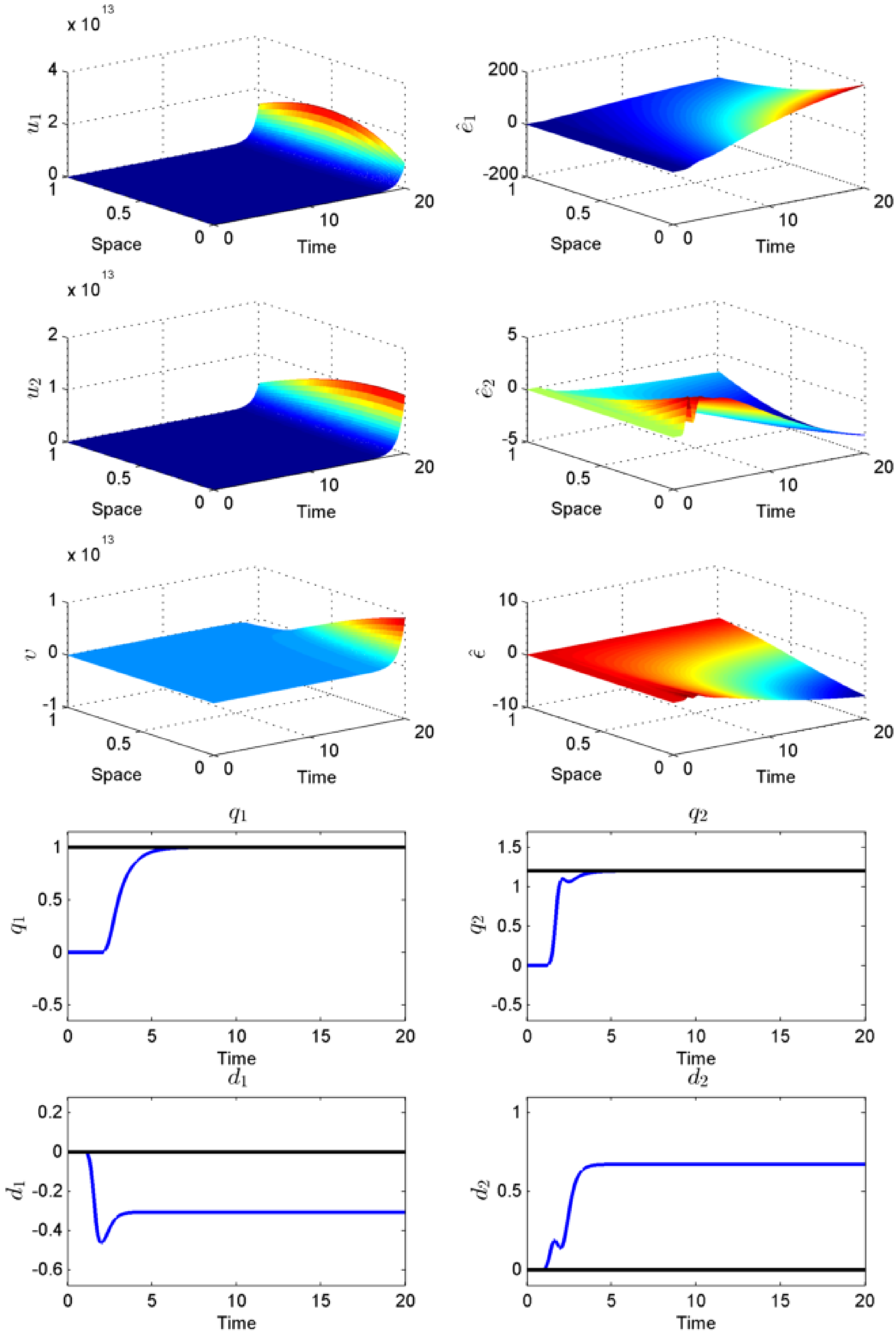

Fig. 6. Estimated states and parameters

f) Part 6:

$$\begin{aligned} &2\int_0^1 e^{\delta x}\mu^{-1}(x)\beta(x,t)\int_0^x h^T(x,\xi)\alpha(\xi,t)d\xi dx \\ &\leq \frac{Mn}{\bar{\mu}}\int_0^1 x e^{\delta x}\beta^2(x,t)dx \\ &+\frac{M}{\bar{\mu}\delta}\int_0^1 (e^{\delta}-e^{\delta x})\alpha^T(x,t)\alpha(x,t)dx \end{aligned} \tag{97}$$

Lumping the different terms together, we obtain (59).

2) Details regarding the bounds on δ and a_1: We must choose $a_1 > 0$ and $\delta > 0$ so that the term in the first parenthesis in (59) and both $\gamma_1(x)$ and $\gamma_2(x)$ given in (60) and (61), respectively, are positive for all $x \in [0,1]$. Firstly, we'll have to choose δ so that the term in the large parenthesis in (60) is positive,

$$\delta > (2+x)\frac{Mn}{\underline{\lambda}} + \frac{Mn}{\underline{\lambda}\delta}. \tag{98}$$

By letting $\delta > 1$, and remembering that $x \in [0,1]$, we obtain

$$\delta > (2+1)\frac{Mn}{\underline{\lambda}} + \frac{Mn}{\underline{\lambda}} = 4\frac{Mn}{\underline{\lambda}}. \tag{99}$$

From (61), we require

$$\delta > \frac{Mn}{\underline{\mu}} + \frac{Mn}{\underline{\mu}}x \tag{100}$$

which yields

$$\delta > 2\frac{Mn}{\underline{\mu}}. \tag{101}$$

Hence, we find the requirement (64) on δ. We also need to choose $a_1 > 0$ so that the term in the first parenthesis in (59) is positive, and so that (60) is positive. From the term in (59), we obtain the requirement

$$a_1 > nM^2 e^{\delta} \tag{102}$$

while from (60), we can omit the positive term $a_1 \frac{Mn}{\underline{\lambda}\delta}e^{-\delta}$ in the analysis, and choose a_1 so that

$$\begin{aligned} &a_1 e^{-\delta x}\left(\delta - (2+x)\frac{Mn}{\underline{\lambda}} - \frac{Mn}{\underline{\lambda}\delta}\right) \\ &\quad - \frac{M}{\underline{\mu}}e^{\delta x} - \frac{M}{\underline{\mu}\delta}e^{\delta} + \frac{M}{\underline{\mu}\delta}e^{\delta x} \end{aligned} \tag{103}$$

is positive. Since, from the choice of δ in (64), the term in the parenthesis is positive we can divide by it, yielding

$$a_1 > \frac{\left(\frac{M}{\underline{\mu}}e^{\delta} + \frac{M}{\underline{\mu}\delta}e^{\delta} - \frac{M}{\underline{\mu}\delta}\right)e^{\delta}}{\delta - 3\frac{Mn}{\underline{\lambda}} - \frac{Mn}{\underline{\lambda}\delta}}. \tag{104}$$

Thus, we find the requirement (65) on a_1, which together with (64) will ensure that $\dot{V}_1$ in (59) is negative semi-definite, concluding the proof.

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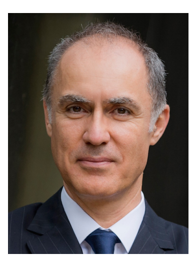

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