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Well-posedness and exact controllability of the mass balance equations for an extrusion process

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In this paper, we study the well-posedness and exact controllability of a physical model for an extrusion process in the isothermal case. The model expresses the mass balance in the extruder chamber and consists of a hyperbolic partial differential equation (PDE) and a nonlinear ordinary differential equation (ODE) whose dynamics describes the evolution of a moving interface. By suitable change of coordinates and fixed point arguments, we prove the existence, uniqueness, and regularity of the solution and finally, the exact controllability of the coupled system. Copyright © 2015 John Wiley & Sons, Ltd.

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1. Introduction

The analysis of free boundary problems has been an active subject in the last decades, and their mathematical understanding continues to be an important interdisciplinary topic for various engineering applications. Representative complex physical systems describing biological phenomena and reaction diffusion processes such as Stefan problems in crystal growth processes are still calling for many open questions related to their exact controllability and the design of highly efficient output feedback control laws for stabilization purposes. Among these challenging problems, one can mention the swelling nanocapsules studied in [1], the lyophilization process applied to pharmaceutical industry [2, 3], the cooking processes describing the volume change in food material [4], the mixing systems (model of torus reactor including a well-mixed zone and a transport zone), and the diesel oxidation catalyst presented in [5].

In this paper, we consider the well-posedness and exact controllability of the Cauchy problem for a physical model of the extrusion process, which describes the mass transport phenomena in an isothermal extruder chamber. Mathematically, the process is described by a hyperbolic PDE defined on a time-varying domain. The dynamics of the spatial domain is governed by an ODE expressing the conservation of mass in the extruder, and the PDE expresses the convection phenomenon due to the rotating screw. More detailed description of the model is given in Section 2. We mention that the first result concerning the mathematical analysis of the extrusion model as transport equations coupled via complementary time-varying domains is proposed in [6], where the well-posedness for the linearized model of the extruder is obtained by using perturbation theory on the linear operator.

The key point is to find a suitable change of coordinates, which enables to transform the free boundary problem to a system defined on a fixed domain. Let us emphasize that this method has been introduced by Li *et al.* to study the free boundary problems of quasilinear hyperbolic systems [7, 8], and now, it is rather standard. Our proof of the well-posedness results (Theorem 3.1) of the Cauchy problem for the normalized system relies on the characteristic method and the fixed point arguments. The only difference is that we consider the (piecewise) H^2 solution (Theorem 3.2) instead of the (piecewise) C^1 solution as in [7, 8], so that the theory on Sobolev spaces can be applied. We remark that the H^2-regularity of the solution is useful when one considers the asymptotic stabilization of the corresponding closed-loop system with feedback controls [9]. In this context, the stabilization of a non-isothermal extrusion process including the temperature and the moisture content dynamics is investigated in [10].

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The control problems for hyperbolic conservation laws have been widely studied for a long time. For controllability of linear hyperbolic systems, one can see the important survey [11]. The controllability of nonlinear hyperbolic equations (or systems) is studied in [12–16]. Moreover, [17] provides a comprehensive survey of controllability and stabilization in PDEs that also includes nonlinear conservation laws. The idea to prove Theorem 3.3 is to construct a solution to (2.2)–(2.4), which also satisfies the final conditions [15]. The way of such construction is based on the controllability result of the linearized system together with fixed point arguments see (for example, [18]).

The organization of this paper is as follows. First, in Section 2, we give a description of the extrusion process model, which is derived from conservation laws. The main results (Theorems 3.1–3.3) concerning the well-posedness, regularity, and the controllability of the normalized system are presented in Section 3, while their proofs are given in Sections 4–6, respectively.

2. Description of the extrusion process model

Extruders are designed to process highly viscous materials. They are mainly used in the chemical industries for polymer processing as well as in the food industries. An extruder is made of a barrel, the temperature of which is regulated. One or two Archimedean screws are rotating inside the barrel. The extruder is equipped with a die where the material comes out of the process (Figure 1).

In an extruder, the net flow at the die exit is mainly due to the flow of the material in the screw axis direction. The die resistance influences highly the transport along the extruder and induces an accumulation phenomenon towards the down barrel direction. The accumulation of the material permits us to represent the mass balance in the extruder using the filling ratio along its screw channel. More precisely, the spatial domain of the extruder might be partitioned into a fully filled zone (FFZ) and a partially filled zone (PFZ). The flow in the FFZ depends on the pressure gradient, which appears in this region because of the die restriction. The PFZ corresponds to a conveying region that is submitted to the constant atmospheric pressure, and the transport velocity of the material depends only on the screw speed and pitch. These two zones are coupled by an interface that is located at the spatial coordinate where the pressure gradient changes from zero to a nonzero value. Basically, the moving interface evolves as a function of the difference between the feed and die rates. In the sequel, the spatial domain of the extruder will be taken as the real interval $[0, L]$, where $L > 0$ is the length of the extruder. Let us denote by $l(t) \in [0, L]$ the position of the thin interface; the domain of the PFZ is then $[0, l(t)]$, and the FFZ is defined on $[l(t), L]$ (Figure 1).

Considering the following change of variables [19]:

$$x \mapsto y = \frac{x}{l(t)} \text{ in PFZ and } x \mapsto y = \frac{x - l(t)}{L - l(t)} \text{ in FFZ} \tag{2.1}$$

respectively, the time-varying domains $[0, l(t)]$ and $[l(t), L]$ can be transformed to the fixed domain $[0, 1]$ in space. For the sake of simplicity, we still denote by x the space variable instead of y. More precisely, we consider the problem for the corresponding normalized system defined on $(0, T) \times (0, 1)$.

Defining the filling ratio along the PFZ spatial domain, namely, $f_p(t, x)$ as a dynamical variable [20, 21], then the mass balance in this region is written as follows:

$$\begin{cases} \partial_t f_p(t,x) + \alpha_p \partial_x f_p(t,x) = 0, & \text{in } \mathbb{R}^+ \times (0,1), \\ f_p(0,x) = f_p^0(x), & \text{in } (0,1), \\ f_p(t,0) = \frac{F_{in}(t)}{\rho_o V_{eff} N(t)}, & \text{in } \mathbb{R}^+, \end{cases} \tag{2.2}$$

where

$$\alpha_p(x, N(t), l(t), f_p(t,1)) = \frac{\zeta N(t) - xF(l(t), N(t), f_p(t,1))}{l(t)} \tag{2.3}$$

and α_p is the transport velocity of the material, ζ is the screw pitch, F_{in} denotes the feed rate, ρ_0 is the melt density, V_{eff} is the effective volume, and $N(t)$ is the rotation speed of the screw. F is the dynamics of the moving interface described by Equations (2.4) and (2.5). The interface motion is generated by the gradient of pressure, which appears in the FFZ [20–23], and under the assumption of constant viscosity along the extruder (the isothermal case), its evolution is given by the following:

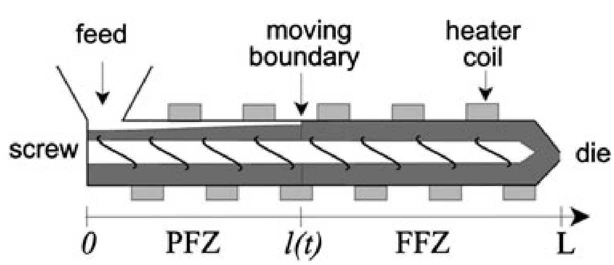

Figure 1. Schematic description of an extruder. FFZ, fully filled zone; PFZ, partially filled zone.

$$\begin{cases} \dot{l}(t) = F(l(t), N(t), f_p(t,1)), & \text{in } \mathbb{R}^+, \\ l(0) = l^0, \end{cases} \tag{2.4}$$

where

$$F(l(t), N(t), f_p(t,1)) = N(t) g(l(t), f_p(t,1)), \tag{2.5}$$

with

$$g(l(t), f_p(t,1)) = \frac{\zeta K_d (L - l(t))}{[B\rho_0 + K_d(L - l(t))](1 - f_p(t,1))} - \frac{\zeta f_p(t,1)}{1 - f_p(t,1)}. \tag{2.6}$$

In (2.6), K_d denotes the die conductance, and B is the geometric parameter.

In the whole paper, unless otherwise specified, we always assume that $l^0 \in (0, L)$, $f_p^0 \in W^{1,\infty}(0,1)$, $F_{in}, N \in L^\infty(0,T)$, $F_{in}/N \in W^{1,\infty}(0,T)$. For the sake of simplicity, we denote from now on $\|f\|_{L^\infty}$ ($\|f\|_{W^{1,\infty}}$, $\|f\|_{L^2}$, resp.) as the L^∞ ($W^{1,\infty}$, L^2, resp.) norm of the function f with respect to its variables.

3. Main results

In this section, we present the main results on the well-posedness and exact controllability of the coupled systems (2.2)–(2.4); we have the following two theorems.

Theorem 3.1
Let $T > 0$ and (l_e, N_e, f_{pe}) be a constant equilibrium, that is,

$$F(l_e, N_e, f_{pe}) = 0 \tag{3.1}$$

with $0 < f_{pe} < 1$, $0 < l_e < L$. Assume that the compatibility condition at $(0,0)$ holds

$$\frac{F_{in}(0)}{\rho_0 V_{eff} N(0)} = f_p^0(0). \tag{3.2}$$

Then, there exists $\varepsilon_0 > 0$ (depending on T) such that for any $\varepsilon \in (0, \varepsilon_0]$, if

$$|l^0 - l_e| + \|f_p^0(\cdot) - f_{pe}\|_{W^{1,\infty}} + \|\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} - f_{pe}\|_{W^{1,\infty}} + \|N(\cdot) - N_e\|_{L^\infty} \leq \varepsilon, \tag{3.3}$$

Cauchy problems (2.2)–(2.4) admit a unique solution $(l, f_p) \in W^{1,\infty}(0,T) \times W^{1,\infty}((0,T)\times(0,1))$, and the following estimates hold:

$$\|l(\cdot) - l_e\|_{W^{1,\infty}} \leq C_{\varepsilon_0} \cdot \varepsilon, \tag{3.4}$$

$$\|f_p(\cdot,\cdot) - f_{pe}\|_{W^{1,\infty}} \leq C_{\varepsilon_0} \cdot \varepsilon, \tag{3.5}$$

where C_{ε_0} is a constant depending on ε_0 but independent of ε.

Theorem 3.2
Under the assumptions of Theorem 3.1, we assume furthermore that $f_p^0(\cdot) \in H^2(0,1)$, $\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} \in H^2(0,T)$, and the compatibility condition at $(0,0)$ holds

$$\left(f_p^0\right)_x(0) + \frac{l(0)}{\zeta N(0)} \cdot \frac{d}{dt}\left(\frac{F_{in}(t)}{\rho_0 V_{eff} N(t)}\right)\Big|_{t=0} = 0. \tag{3.6}$$

Then, there exists $\varepsilon_0 > 0$ (depending on T) such that for any $\varepsilon \in (0, \varepsilon_0]$, if

$$|l^0 - l_e| + \|f_p^0(\cdot) - f_{pe}\|_{H^2(0,1)} + \|\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} - f_{pe}\|_{H^2(0,T)} + \|N(\cdot) - N_e\|_{L^\infty} \leq \varepsilon, \tag{3.7}$$

Cauchy problems (2.2)–(2.4) have a unique solution $(l, f_p) \in W^{1,\infty}(0,T) \times C^0([0,T]; H^2(0,1))$ with the additional estimate

$$\|f_p(\cdot,\cdot) - f_{pe}\|_{C^0([0,T];H^2(0,1))} \leq C_{\varepsilon_0} \cdot \varepsilon, \tag{3.8}$$

where C_{ε_0} is a constant depending on ε_0 but independent of ε.

Remark 3.1
The solution in Theorem 3.1 or in Theorem 3.2 is often called semi-global solution because it exists on any preassigned time interval $[0, T]$ provided that (l, f_p) has some kind of smallness (depending on T); see [24, 25].

Remark 3.2
We have the hidden regularity that $f_p \in C^0([0,1]; H^2(0,T))$ in Theorem 3.2 (see [26, 27] for the idea of proof).

The problem of exact controllability for Cauchy problems (2.2)–(2.4) can be described as follows: for any given initial data $(l^0, f_p^0(x))$, any final data $(l^1, f_p^1(x))$, to find a time T and controls $F_{in}(t)$ and $N(t)$ such that the solution to Cauchy problems (2.2)–(2.4) satisfies

$$l(T) = l^1, \tag{3.9}$$

$$f_p(T, x) = f_p^1(x). \tag{3.10}$$

Our result is the following theorem on local controllability in the sense that the initial and final data are both close to the given equilibrium determined by (3.1).

Theorem 3.3
Let

$$T_e := \frac{l_e}{\zeta N_e} \tag{3.11}$$

be the critical control time. Then, for any $T > T_e$, there exists $\nu_1 > 0$ suitably small such that, for any $\nu \in (0, \nu_1]$, $l^0, l^1 \in (0, L)$, and $f_p^0, f_p^1 \in W^{1,\infty}(0,1)$ with

$$|l^0 - l_e| + |l^1 - l_e| + \|f_p^0(\cdot) - f_{pe}\|_{W^{1,\infty}} + \|f_p^1(\cdot) - f_{pe}\|_{W^{1,\infty}} \le \nu, \tag{3.12}$$

there exist $N \in L^\infty(0,T)$ and $F_{in} \in L^\infty(0,T)$ satisfying

$$\left\|\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} - f_{pe}\right\|_{W^{1,\infty}} + \|N(\cdot) - N_e\|_{L^\infty} \le C_{\nu_1} \cdot \nu, \tag{3.13}$$

such that the weak solution $(l(t), f_p(t,x))$ to Cauchy problems (2.2)–(2.4) satisfies the final conditions (3.9) and (3.10). Here, C_{ν_1} is a constant depending on ν_1 but independent of ν.

4. Proof of Theorem 3.1

In order to conclude Theorem 3.1, it suffices to prove the following lemma on local well-posedness of Cauchy problems (2.2)–(2.4).

Lemma 4.1
There exist $\varepsilon_1 > 0$ and $\delta > 0$ suitably small, such that for any $\varepsilon \in (0, \varepsilon_1]$, $l^0 \in (0, L)$, $f_p^0 \in W^{1,\infty}(0,1)$, $F_{in}/N \in W^{1,\infty}(0,T)$ with

$$|l^0 - l_e| + \|f_p^0(\cdot) - f_{pe}\|_{W^{1,\infty}} + \left\|\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} - f_{pe}\right\|_{W^{1,\infty}} + \|N(\cdot) - N_e\|_{L^\infty} \le \varepsilon, \tag{4.1}$$

Cauchy problems (2.2)–(2.4) admit a unique local solution on $[0, \delta]$, which satisfies the following estimates:

$$|l(t) - l_e| \le C_{\varepsilon_1} \cdot \varepsilon, \quad \forall t \in [0, \delta], \tag{4.2}$$

$$\|f_p(t,\cdot) - f_{pe}\|_{W^{1,\infty}} \le C_{\varepsilon_1} \cdot \varepsilon, \quad \forall t \in [0, \delta], \tag{4.3}$$

where C_{ε_1} is a constant depending on ε_1 but independent of ε.

Let us first show how to conclude Theorem 3.1 from Lemma 4.1. By Lemma 4.1, we take $\varepsilon_2 \in (0, \varepsilon_1]$ such that $C_{\varepsilon_1} \cdot \varepsilon_2 \le \varepsilon_1$. Then for any $\varepsilon \in (0, \varepsilon_2]$ and any initial-boundary data such that (4.1) holds, Cauchy problems (2.2)–(2.4) admit a unique solution on $[0, \delta]$. Furthermore, one has

$$|l(\delta) - l_e| \le C_{\varepsilon_1} \cdot \varepsilon \le \varepsilon_1, \tag{4.4}$$

$$\|f_p(\delta, \cdot) - f_{pe}\|_{W^{1,\infty}} \le C_{\varepsilon_1} \cdot \varepsilon \le \varepsilon_1. \tag{4.5}$$

By taking $(l(\delta), f_p(\delta,\cdot))$ as new initial data and applying Lemma 4.1 on $[\delta, 2\delta]$, the solution of Cauchy problems (2.2)–(2.4) is extended to $[0, 2\delta]$. For fixed $T > 0$, we can extend the local solution to Cauchy problems (2.2)–(2.4) to $[0, T]$ eventually by reducing the value of ε_0 and applying Lemma 4.1 in finite times (at most $[T/\delta] + 1$ times). Therefore, to conclude Theorem 3.1, it remains to prove Lemma 4.1.

Proof of Lemma 4.1

The proof is divided into four steps.

Step 1. Existence and uniqueness of $(l(\cdot), f_p(\cdot, 1))$ by fixed point argument.

Let $\varepsilon_1 > 0$ be such that

$$0 < \varepsilon_1 < \min\{l_e, L - l_e, f_{pe}, 1 - f_{pe}\}. \tag{4.6}$$

Denote

$$\|F\|_{W^{1,\infty}} := \sum_{|\alpha|\le 1} \operatorname*{ess\,sup}_{\substack{0<x_1<L \\ N_e-\varepsilon_1<x_2<N_e+\varepsilon_1 \\ 0<x_3<1}} |D^\alpha F(x_1, x_2, x_3)|, \tag{4.7}$$

$$\Psi(t) := (l(t), f_p(t, 1)), \quad t \in [0, T]. \tag{4.8}$$

For any given $\delta > 0$ small enough (to be chosen later), we define a domain candidate as a closed subset of $C^0([0, \delta])$ with respect to C^0 norm:

$$\Omega_{\delta,\varepsilon_1} := \left\{\Psi \in C^0([0, \delta]) | \Psi(0) = \left(l^0, f_p^0(1)\right),\ \|\Psi(\cdot) - (l_e, f_{pe})\|_{C^0([0,\delta])} \le \varepsilon_1\right\}. \tag{4.9}$$

We denote by $\xi(s; t, x)$, with $(s, \xi(s; t, x)) \in [0, t] \times [0, 1]$, the characteristic curve passing through the point $(t, x) \in [0, \delta] \times [0, 1]$ (Figure 2), that is,

$$\begin{cases} \dfrac{d\xi(s; t, x)}{ds} = \alpha_p\left(\xi(s; t, x), N(s), l(s), f_p(s, 1)\right), \\ \xi(t; t, x) = x. \end{cases} \tag{4.10}$$

Let us define a map $\mathfrak{F} := (\mathfrak{F}_1, \mathfrak{F}_2)$, where $\mathfrak{F} : \Omega_{\delta,\varepsilon_1} \to C^0([0, \delta])$, $\Psi \mapsto \mathfrak{F}(\Psi)$ as

$$\mathfrak{F}_1(\Psi)(t) := l^0 + \int_0^t F\left(l(s), N(s), f_p(s, 1)\right) ds, \tag{4.11}$$

$$\mathfrak{F}_2(\Psi)(t) := f_p^0(\xi(0; t, 1)). \tag{4.12}$$

Solving the linear ODE (4.10) with α_p given by (2.3), one easily obtains for all δ small and all $0 \le s \le t \le \delta$ that

$$\xi(s; t, 1) = e^{\int_s^t \frac{F(l(\sigma),N(\sigma),f_p(\sigma,1))}{l(\sigma)} d\sigma} - \int_s^t \frac{\zeta N(\sigma)}{l(\sigma)} \cdot e^{\int_s^\sigma \frac{F(l(s),N(s),f_p(s,1))}{l(s)} ds} d\sigma. \tag{4.13}$$

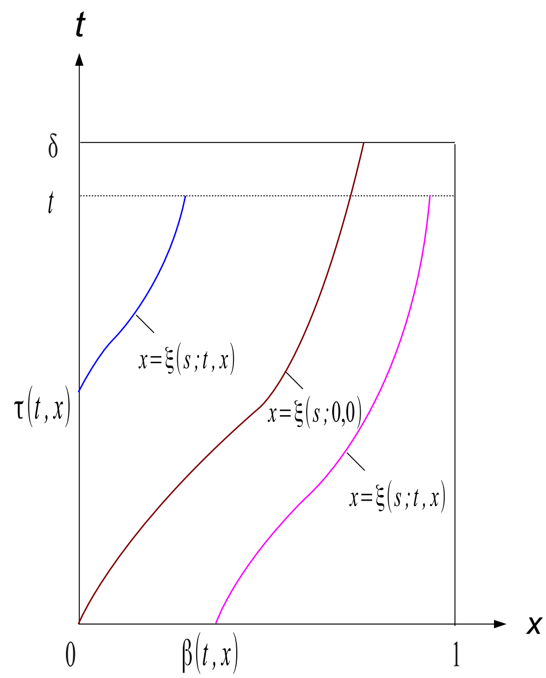

Figure 2. The characteristics $\xi(s; t, x)$ and $\beta(t, x)$, $\tau(t, x)$.

It is obvious that $\mathfrak{F}$ maps into $\Omega_{\delta,\varepsilon_1}$ itself if

$$0<\delta<\min\left\{T,\frac{l_e-\varepsilon_1}{\zeta(N_e+\varepsilon_1)},\frac{l_e-\varepsilon_1}{\|F\|_{W^{1,\infty}}},\frac{L-l_e-\varepsilon_1}{\|F\|_{W^{1,\infty}}}\right\}, \tag{4.14}$$

Now, we prove that, if δ is small enough, $\mathfrak{F}$ is a contraction mapping on $\Omega_{\delta,\varepsilon_1}$ with respect to the C^0 norm. Let $\Psi=(l,f_p)$, $\bar\Psi=(\bar l,\bar f_p)\in\Omega_{\delta,\varepsilon_1}$. We denote by $\bar\xi(s;t,x)$ the corresponding characteristic curve passing through (t,x):

$$\begin{cases}\dfrac{d\bar\xi(s;t,x)}{ds}=\alpha_p\left(\bar\xi(s;t,x),N(s),\bar l(s),\bar f_p(s,1)\right),\\ \bar\xi(t;t,x)=x.\end{cases} \tag{4.15}$$

Similarly as (4.13), one has for all δ small and all $0\le s\le t\le\delta$ that

$$\bar\xi(s;t,1)=e^{\int_s^t\frac{F(\bar l(\sigma),N(\sigma),\bar f_p(\sigma,1))}{\bar l(\sigma)}\,d\sigma}-\int_s^t\frac{\zeta N(\sigma)}{\bar l(\sigma)}\cdot e^{\int_s^\sigma\frac{F(\bar l(s),N(s),\bar f_p(s,1))}{\bar l(s)}\,ds}\,d\sigma. \tag{4.16}$$

Therefore, it holds for all $t\in[0,\delta]$ that

$$\begin{aligned}|\mathfrak{F}_1(\bar\Psi)(t)-\mathfrak{F}_1(\Psi)(t)|&=\left|\int_0^t F(\bar l(s),N(s),\bar f_p(s,1))\,ds-\int_0^t F(l(s),N(s),f_p(s,1))\,ds\right|\\&\le\delta\|F\|_{W^{1,\infty}}\|\bar\Psi-\Psi\|_{C^0([0,\delta])}.\end{aligned} \tag{4.17}$$

On the other hand, it follows from (4.12), (4.13), and (4.16) that for all $t\in[0,\delta]$,

$$\begin{aligned}&|\mathfrak{F}_2(\bar\Psi)(t)-\mathfrak{F}_2(\Psi)(t)|\\&=|f_p^0(\bar\xi(0;t,1))-f_p^0(\xi(0;t,1))|\\&\le\|f_{px}^0\|_{L^\infty}\left(\left|e^{\int_0^t\frac{F(\bar l(\sigma),N(\sigma),\bar f_p(\sigma,1))}{\bar l(\sigma)}\,d\sigma}-e^{\int_0^t\frac{F(l(\sigma),N(\sigma),f_p(\sigma,1))}{l(\sigma)}\,d\sigma}\right|\right.\\&\qquad\left.+\int_0^t\left|\frac{\zeta N(\sigma)}{\bar l(\sigma)}\cdot e^{\int_0^\sigma\frac{F(\bar l(s),N(s),\bar f_p(s,1))}{\bar l(s)}\,ds}-\frac{\zeta N(\sigma)}{l(\sigma)}\cdot e^{\int_0^\sigma\frac{F(l(s),N(s),f_p(s,1))}{l(s)}\,ds}\right|\,d\sigma\right).\end{aligned}$$

By (4.7) and the fact that $\Psi,\bar\Psi\in\Omega_{\delta,\varepsilon_1}$ and $l(t)\ge l_e-\varepsilon_1>0$, $|N(t)|\le N_e+\varepsilon_1$, it follows that for all $t\in[0,\delta]$,

$$|\mathfrak{F}_2(\bar\Psi)(t)-\mathfrak{F}_2(\Psi)(t)|\le C\delta\|f_p^0(\cdot)-f_{pe}\|_{W^{1,\infty}}\|\bar\Psi-\Psi\|_{C^0([0,\delta])}, \tag{4.18}$$

where $C>0$ is a constant independent of $(\bar\Psi,\Psi)$. Finally, combining (4.17) and (4.18), we can choose δ small enough such that

$$\|\mathfrak{F}(\bar\Psi)-\mathfrak{F}(\Psi)\|_{C^0([0,\delta])}\le\frac{1}{2}\|\bar\Psi-\Psi\|_{C^0([0,\delta])}. \tag{4.19}$$

Banach fixed point theorem implies the existence of the unique fixed point $(l(\cdot),f_p(\cdot,1))$ of the mapping $\mathfrak{F}$: $\Psi=\mathfrak{F}(\Psi)$ in $\Omega_{\delta,\varepsilon_1}$.

Step 2. Construction of a solution by characteristic method.

With the existence of $(l(\cdot),f_p(\cdot,1))$ and $\delta>0$ by step 1, we can construct a solution to Cauchy problems (2.2)–(2.4). For every (t,x) in $[0,\delta]\times[0,1]$, we still denote by $\xi(s;t,x)$, with $(s,\xi(s;t,x))\in[0,t]\times[0,1]$, the characteristic curve passing through the point (t,x); see (4.10). Because the velocity function α_p is positive, the characteristic $\xi(s;t,x)$ intersects the x-axis at point $(0,\beta(t,x))$ with $\beta(t,x)=\xi(0;t,x)$ if $0\le\xi(t;0,0)\le x\le 1$; the characteristic $\xi(s;t,x)$ intersects the t-axis at point $(\tau(t,x),0)$ with $\xi(\tau(t,x);t,x)=0$ if $0\le x\le\xi(t;0,0)$ (Figure 2). Moreover, we have (see [8, Lemma 3.2 and its proof, pp. 90–91] for a more general situation)

$$\frac{\partial\tau(t,x)}{\partial x}=\frac{-l(\tau(t,x))}{\zeta N(\tau(t,x))}\,e^{\int_{\tau(t,x)}^t\frac{F(l(\sigma),N(\sigma),f_p(\sigma,1))}{l(\sigma)}\,d\sigma}, \tag{4.20}$$

$$\frac{\partial\beta(t,x)}{\partial x}=e^{\int_0^t\frac{F(l(\sigma),N(\sigma),f_p(\sigma,1))}{l(\sigma)}\,d\sigma}. \tag{4.21}$$

We define f_p by

$$f_p(t,x)=\begin{cases}\dfrac{F_{in}(\tau(t,x))}{\rho_0V_{eff}N(\tau(t,x))}, & \text{if } 0\le x\le\xi(t;0,0)\le 1, 0\le t\le\delta,\\ f_p^0(\beta(t,x)), & \text{if } 0\le\xi(t;0,0)\le x\le 1, 0\le t\le\delta.\end{cases} \tag{4.22}$$

Then it is easy to check that $(l, f_p) \in W^{1,\infty}(0,\delta) \times W^{1,\infty}((0,\delta) \times (0,1))$ under the compatibility condition (3.2), and (l, f_p) is indeed a solution to Cauchy problems (2.2)–(2.4).

Step 3. Uniqueness of the solution.

Assume that Cauchy problems (2.2)–(2.4) have two solutions (l, f_p), $(\bar{l}, \bar{f}_p)$ on $[0,\delta] \times [0,1]$. It follows that $(l(\cdot), f_p(\cdot,1)) = (\bar{l}(\cdot), \bar{f}_p(\cdot,1))$ because they are both the fixed point of the mapping $\mathfrak{F}$: $\Psi = \mathfrak{F}(\Psi)$ in $\Omega_{\delta,\varepsilon_1}$. This fact implies that the characteristics $\xi(\cdot;t,x)$ and $\bar{\xi}(\cdot;t,x)$ coincide with each other and therefore so do the solutions f_p and $\bar{f}_p$ by characteristic method.

Step 4. A priori estimate on the local solution.

By definition of f_p and assumption (4.1), it is clear that for all $t \in [0,\delta]$,

$$\|f_p(t,\cdot) - f_{pe}\|_{L^\infty} \le \varepsilon. \tag{4.23}$$

Thanks to (2.4), (3.1), (4.23), and assumption (4.1), we obtain for all $t \in [0,\delta]$ that

$$\begin{aligned}|\dot{l}(t)| =& |F\left(l(t), N(t), f_p(t,1)\right) - F\left(l_e, N_e, f_{pe}\right)| \\ \le& \|F\|_{W^{1,\infty}}(|l(t) - l_e| + |N(t) - N_e| + |f_p(t,1) - f_{pe}|). \\ \le& \|F\|_{W^{1,\infty}}|l(t) - l_e| + 2\varepsilon\|F\|_{W^{1,\infty}},\end{aligned}$$

which yields (4.2) from (4.1) and Gronwall's inequality. On the other hand, from (4.22),

$$\begin{aligned}\left\|\frac{\partial f_p}{\partial x}\right\|_{L^\infty} \le& \left\|\frac{\partial}{\partial x}\left(\frac{F_{in}(\tau(t,x))}{\rho_0 V_{eff} N(\tau(t,x))}\right)\right\|_{L^\infty} + \left\|\frac{\partial}{\partial x} f_p^0(\beta(t,x))\right\|_{L^\infty} \\ \le& \left\|\frac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} - f_{pe}\right\|_{W^{1,\infty}} \left\|\frac{\partial \tau}{\partial x}\right\|_{L^\infty} + \left\|f_p^0(\cdot) - f_{pe}\right\|_{W^{1,\infty}} \left\|\frac{\partial \beta}{\partial x}\right\|_{L^\infty}.\end{aligned} \tag{4.24}$$

Combining (4.20), (4.21), (4.24), and assumption (4.1), we obtain (4.3), which concludes the proof of Lemma 4.1. □

5. Proof of Theorem 3.2

Before proving Theorem 3.2, let us recall a classical result on Cauchy problem of the following general linear transport equation:

$$\begin{cases} u_t + a(t,x)u_x = b(t,x)u + c(t,x), & (t,x) \in (0,T) \times (0,1), \\ u(0,x) = u_0(x), & x \in (0,1), \\ u(t,0) = h(t), & t \in (0,T), \end{cases} \tag{5.1}$$

where $a(t,x) > 0$, $a, a_x, b \in L^\infty((0,T) \times (0,1))$, and $c \in L^2((0,T) \times (0,1))$.

We recall from [17, Section 2.1] the definition of a weak solution to Cauchy problem (5.1).

Definition 5.1

Let $T > 0$, $u_0 \in L^2(0,1)$, and $h \in L^2(0,T)$ be given. A weak solution of Cauchy problem (5.1) is a function $u \in C^0([0,T]; L^2(0,1))$ such that for every $\tau \in [0,T]$, every test function $\varphi \in C^1([0,T] \times [0,1])$ such that $\varphi(t,1) = 0,\ \forall t \in [0,T]$, one has

$$\begin{aligned}&-\int_0^\tau \int_0^1 (u[\partial_t\varphi + a\partial_x\varphi + (a_x + b)\varphi] + c\varphi)\, dx\, dt + \int_0^1 u(\tau,\cdot)\varphi(\tau,\cdot)\, dx \\ &-\int_0^1 u_0\varphi(0,\cdot)\, dx - \int_0^\tau ha(\cdot,0)\varphi(\cdot,0)\, dt = 0.\end{aligned} \tag{5.2}$$

We have the following lemma.

Lemma 5.1

Let $T > 0$, $u_0 \in L^2(0,1)$, and $h \in L^2(0,T)$ be given. Then, Cauchy problem (5.1) has a unique weak solution $u \in C^0([0,T]; L^2(0,1))$, and the following estimate holds:

$$\|u\|_{C^0([0,T];L^2(0,1))} \le C(\|u_0\|_{L^2(0,1)} + \|h\|_{L^2(0,T)} + \|c\|_{L^2((0,T)\times(0,1))}), \tag{5.3}$$

where $C = C(T, \|a\|_{L^\infty((0,T)\times(0,1))}, \|a_x\|_{L^\infty((0,T)\times(0,1))}, \|b\|_{L^\infty((0,T)\times(0,1))})$ is a constant independent of u_0, h, c.

For the proof of Lemma 5.1, one can refer to [8] for classical solution or [28, Theorem 23.1.2, Page 387] for Cauchy problem on $\mathbb{R}$ without boundary.

Proof of Theorem 3.2

By Theorem 3.1 and Lemma 5.1, it suffices to prove that the systems of $f_{p_{xx}}$ satisfy all the assumptions of Lemma 5.1.

Differentiating (2.2) with respect to x once and twice gives us successively that

$$\begin{cases} \partial_t f_{p_x}(t,x) + \alpha_p(t,x)\partial_x f_{p_x}(t,x) = -\alpha_{p_x}(t,x) f_{p_x}(t,x), & \text{in } (0,T)\times(0,1), \\ f_{p_x}(0,x) = f^0_{p_x}(x), & \text{in } (0,1), \\ f_{p_x}(t,0) = \dfrac{-l(t)}{\zeta N(t)} \cdot \dfrac{d}{dt}\left(\dfrac{F_{in}(t)}{\rho_0 V_{eff} N(t)}\right), & \text{in } (0,T), \end{cases} \tag{5.4}$$

and

$$\begin{cases} \partial_t f_{p_{xx}}(t,x) + \alpha_p(t,x)\partial_x f_{p_{xx}}(t,x) = -2\alpha_{p_x}(t,x) f_{p_{xx}}(t,x), & \text{in } (0,T)\times(0,1), \\ f_{p_{xx}}(0,x) = f^0_{p_{xx}}(x), & \text{in } (0,1), \\ f_{p_{xx}}(t,0) = \dfrac{-l(t)}{\zeta N(t)}\left[\dfrac{F(l(t),N(t),f_p(t,1))}{\zeta N(t)} \cdot \dfrac{d}{dt}\left(\dfrac{F_{in}(t)}{\rho_0 V_{eff} N(t)}\right)\right. & \\ \qquad\qquad \left. -\dfrac{d}{dt}\left(\dfrac{l(t)}{\zeta N(t)} \cdot \dfrac{d}{dt}\left(\dfrac{F_{in}(t)}{\rho_0 V_{eff} N(t)}\right)\right)\right], & \text{in } (0,T), \end{cases} \tag{5.5}$$

with

$$\alpha_p(t,x) = \frac{\zeta N(t) - xF(l(t),N(t),f_p(t,1))}{l(t)}, \quad \alpha_{p_x}(t,x) = \frac{-F(l(t),N(t),f_p(t,1))}{l(t)}. \tag{5.6}$$

From the assumptions that $f^0_p \in H^2(0,1)$, $\dfrac{F_{in}(\cdot)}{\rho_0 V_{eff} N(\cdot)} \in H^2(0,T)$ and the compatibility conditions (3.2) and (3.6), one easily concludes Theorem 3.2 by applying Lemma 5.1 to Cauchy problem (5.5). □

6. Proof of Theorem 3.3

The idea to prove Theorem 3.3 is to construct a solution to Cauchy problems (2.2)–(2.4), which also satisfies the final conditions. The way of such construction is based on the controllability result of the linearized system together with fixed point arguments [18].

For any fixed initial data $\left(l^0, f^0_p(x)\right)$ and final data $\left(l^1, f^1_p(x)\right)$ close to the equilibrium (l_e, f_{pe}), we define a domain as a closed subset of $C^0([0,T])$ with respect to C^0 norm:

$$\Omega^{\varepsilon_1,T} := \left\{\Phi \in C^0([0,T]) | \Phi(0) = \left(l^0, f^0_p(1)\right), \Phi(T) = \left(l^1, f^1_p(1)\right), \|\Phi(\cdot) - (l_e, f_{pe})\|_{C^0([0,T])} \le \varepsilon_1\right\},$$

where the constant $\varepsilon_1 > 0$ is determined by (4.6). We study the exact controllability for the linearized system deduced from Equations (2.2)–(2.4), replacing $(l(t), f_p(t,1))$ by $(a(t), b(t))$ in functions F and α_p:

$$\begin{cases} \dot{l}(t) = F^{a,b}(t), & \text{in } (0,T), \\ \partial_t f_p(t,x) + \alpha^{a,b}_p(t,x)\partial_x f_p(t,x) = 0, & \text{in } (0,T)\times(0,1), \\ l(0) = l^0, \quad f_p(0,x) = f^0_p(x), & \text{in } (0,1), \\ f_p(t,0) = \dfrac{F^{a,b}_{in}(t)}{\rho_o V_{eff} N^{a,b}(t)}, & \text{in } (0,T), \end{cases} \tag{6.1}$$

where

$$F^{a,b}(t) = N^{a,b}(t) g^{a,b}(t), \tag{6.2}$$

$$g^{a,b}(t) = \frac{\zeta K_d (L - a(t))}{[B\rho_0 + K_d(L - a(t))](1 - b(t))} - \frac{\zeta b(t)}{1 - b(t)}, \tag{6.3}$$

$$\alpha^{a,b}_p(t,x) = \frac{\zeta N^{a,b}(t) - xF^{a,b}(t)}{a(t)}. \tag{6.4}$$

By assumption $T > T_e$, it is possible to find controls $N^{a,b}(t)$ and $F^{a,b}_{in}(t)$ such that the solution of (6.1) satisfies (3.9) and (3.10).

For any $\Phi = (a,b) \in \Omega^{\varepsilon_1,T}$, we choose the control function $N^{a,b}(t)$ as

$$N^{a,b}(t) := \frac{l^1 - l^0}{T} \cdot \frac{1}{g^{a,b}(t)}, \quad t \in [0,T], \tag{6.5}$$

such that

$$F^{a,b}(t) = \frac{l^1 - l^0}{T}, \quad t \in [0,T], \tag{6.6}$$

thus, $l(t)$ is a linear function of t:

$$l(t) = l^0 + \frac{l^1 - l^0}{T} t, \quad t \in [0, T]. \tag{6.7}$$

By assumption (3.12), it follows then

$$|l(t) - l_e| = \left| \left(1 - \frac{t}{T}\right)(l^0 - l_e) + \frac{t}{T}(l^1 - l_e) \right| \le 2\nu \le 2\nu_1. \tag{6.8}$$

Next, let us construct the desired control $F_{in}^{a,b}(t)$. For this purpose, we shall give the expression of $f_p(t,x)$ by characteristic method. We denote by $\xi^{a,b}(s;t,x)$ the characteristic passing through $(t,x) \in [0,T] \times [0,1]$:

$$\begin{cases} \dfrac{d\xi^{a,b}(s;t,x)}{ds} = \alpha_p^{a,b}(s, \xi^{a,b}(s;t,x)) = \dfrac{\zeta N^{a,b}(s) - \xi^{a,b}(s;t,x) F^{a,b}(s)}{a(s)}, \\ \xi^{a,b}(t;t,x) = x. \end{cases} \tag{6.9}$$

Suppose that the characteristic $\xi^{a,b}(s;0,0)$ intersects the line $x = 1$ at $(t_0^{a,b}, 1)$. Then for every $t \in [0, t_0^{a,b}]$, the characteristic passing through $(t,1)$ intersects with x-axis at $(0, \beta^{a,b}(t))$; for every $t \in (t_0^{a,b}, T]$, the characteristic passing through $(t,1)$ intersects with t-axis at $(\tau^{a,b}(t), 0)$. The characteristic passing through $(T,1)$ intersects with t-axis at $(t_1^{a,b}, 0)$ (Figure 3). Hence, we have

$$\xi^{a,b}\left(t_0^{a,b}; 0, 0\right) = 1, \quad \xi^{a,b}(0;t,1) = \beta^{a,b}(t), \quad \xi^{a,b}(\tau^{a,b}(t);t,1) = 0, \quad \xi^{a,b}\left(t_1^{a,b}; T, 1\right) = 0.$$

Solving (6.9) and using (6.4) and (6.6), we obtain

$$e^{\int_0^{t_0^{a,b}} \frac{l^1 - l^0}{Ta(\sigma)} d\sigma} - \int_0^{t_0^{a,b}} \frac{\zeta N^{a,b}(\sigma)}{a(\sigma)} \cdot e^{\int_0^{\sigma} \frac{l^1 - l^0}{Ta(s)} ds} d\sigma = 1, \tag{6.10}$$

$$e^{\int_0^{t} \frac{l^1 - l^0}{Ta(\sigma)} d\sigma} - \int_0^{t} \frac{\zeta N^{a,b}(\sigma)}{a(\sigma)} \cdot e^{\int_0^{\sigma} \frac{l^1 - l^0}{Ta(s)} ds} d\sigma = \beta^{a,b}(t), \tag{6.11}$$

$$e^{\int_{\tau^{a,b}(t)}^{t} \frac{l^1 - l^0}{Ta(\sigma)} d\sigma} - \int_{\tau^{a,b}(t)}^{t} \frac{\zeta N^{a,b}(\sigma)}{a(\sigma)} \cdot e^{\int_{\tau^{a,b}(t)}^{\sigma} \frac{l^1 - l^0}{Ta(s)} ds} d\sigma = 1, \tag{6.12}$$

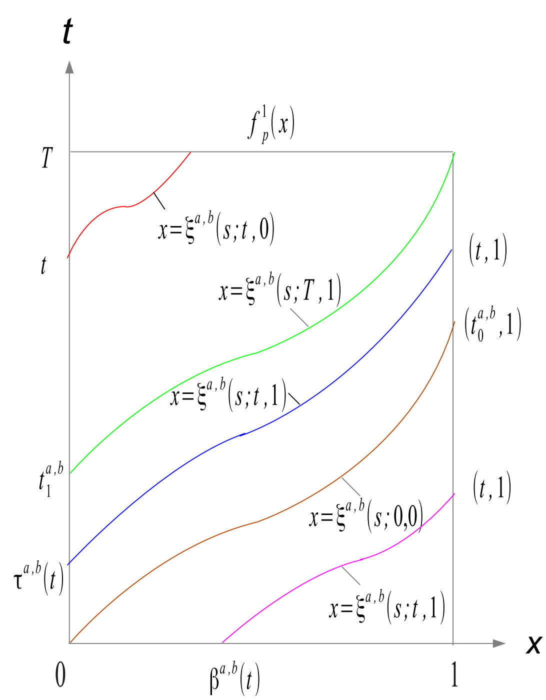

Figure 3. The characteristics $\xi^{a,b}(s;t,x)$ and $t_0^{a,b}$, $\beta^{a,b}(t)$, $\tau^{a,b}(t)$, $t_1^{a,b}$.

$$e^{\int_{t_1^{a,b}}^{T}\frac{l^1-l^0}{Ta(\sigma)}\,d\sigma}-\int_{t_1^{a,b}}^{T}\frac{\zeta N^{a,b}(\sigma)}{a(\sigma)}\cdot e^{\int_{t_1^{a,b}}^{\sigma}\frac{l^1-l^0}{Ta(s)}\,ds}\,d\sigma=1. \tag{6.13}$$

We define the control function $F_{in}^{a,b}(t)$ as follows:

$$F_{in}^{a,b}(t)=\begin{cases} h(t)\rho_0 V_{eff}N^{a,b}(t), & t\in[0,t_1^{a,b}],\\ f_p^1(\xi^{a,b}(T;t,0))\rho_0 V_{eff}N^{a,b}(t), & t\in\left[t_1^{a,b},T\right],\end{cases} \tag{6.14}$$

where $h(t)$ is any artificial $W^{1,\infty}$ function satisfying

$$\|h(\cdot)-f_{pe}\|_{W^{1,\infty}}\le\nu_1 \tag{6.15}$$

and the following compatibility conditions:

$$h(0)=f_p^0(0),\quad h\left(t_1^{a,b}\right)=f_p^1(1). \tag{6.16}$$

Inspired by (4.22), we define a map $\mathcal{F}:=(\mathcal{F}_1,\mathcal{F}_2):\Omega^{\varepsilon_1,T}\to C^0([0,T]),\ \Phi\mapsto\mathcal{F}(\Phi)$ as

$$\mathcal{F}_1(\Phi)(t):=l(t)=l_0+\frac{l^1-l^0}{T}t,\quad t\in[0,T], \tag{6.17}$$

$$\mathcal{F}_2(\Phi)(t):=\begin{cases} f_p^0(\beta^{a,b}(t)),\ t\in\left[0,t_0^{a,b}\right),\\ h(\tau^{a,b}(t)),\ t\in\left(t_0^{a,b},T\right],\end{cases} \tag{6.18}$$

where $t_0^{a,b}$, $\beta^{a,b}(t)$, and $\tau^{a,b}(t)$ are defined by (6.10), (6.11), and (6.12), respectively.

Now, we prove that $\mathcal{F}$ is a contraction mapping on $\Omega^{\varepsilon_1,T}$ provided that $\nu_1>0$ is small. Obviously, the existence of the fixed point implies the existence of the desired control to the original nonlinear controllability problem.

Let $\nu_1\le\varepsilon_1/2$. By (3.12), (6.8), (6.15), (6.17), and (6.18), $\mathcal{F}(\Phi)\in\Omega^{\varepsilon_1,T}$ for all $\Phi\in\Omega^{\varepsilon_1,T}$.

For any $\bar\Phi=(\bar a,\bar b)\in\Omega^{\varepsilon_1,T}$, we denote by $\xi^{\bar a,\bar b}(s;t,x)$ the characteristic passing through $(t,x)\in[0,T]\times[0,1]$ as in (6.9) upon replacing a,b by $\bar a,\bar b$. Correspondingly, we define $t_0^{\bar a,\bar b}$, $\beta^{\bar a,\bar b}(t)$, $\tau^{\bar a,\bar b}(t)$, and $t_1^{\bar a,\bar b}$ as in (6.10), (6.11), (6.12), and (6.13) upon replacing a,b by $\bar a,\bar b$.

By definition of $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2)$ in (6.17) and (6.18), we obtain $\mathcal{F}_1(\bar\Phi)\equiv\mathcal{F}_1(\Phi)$, and thus,

$$\|\mathcal{F}(\bar\Phi)-\mathcal{F}(\Phi)\|_{C^0([0,T])}=\|\mathcal{F}_2(\bar\Phi)-\mathcal{F}_2(\Phi)\|_{C^0([0,T])}=\sup_{t\in[0,T]}|(\mathcal{F}_2(\bar\Phi)-\mathcal{F}_2(\Phi))(t)|.$$

Without loss of generality, we may assume that $t_0^{\bar a,\bar b}>t_0^{a,b}$. Hence, we need to estimate the point-wisely $|\mathcal{F}_2(\bar\Phi)(t)-\mathcal{F}_2(\Phi)(t)|$ on the time interval $\left[0,t_0^{a,b}\right]$, $\left[t_0^{a,b},t_0^{\bar a,\bar b}\right]$, and $\left[t_0^{\bar a,\bar b},T\right]$, respectively.

For any given $t\in\left[0,t_0^{a,b}\right]$, by (3.12) and (6.18), we have

$$\begin{aligned}|(\mathcal{F}_2(\bar\Phi)-\mathcal{F}_2(\Phi))(t)|&=\left|f_p^0(\beta^{\bar a,\bar b}(t))-f_p^0(\beta^{a,b}(t))\right|\\&\le\|f_{px}^0\|_{L^\infty}|\beta^{\bar a,\bar b}(t)-\beta^{a,b}(t)|\\&\le\nu_1|\beta^{\bar a,\bar b}(t)-\beta^{a,b}(t)|.\end{aligned}$$

By (6.11), it is easy to obtain

$$|\beta^{\bar a,\bar b}(t)-\beta^{a,b}(t)|\le C\left(\|\bar a-a\|_{C^0([0,T])}+\|\bar b-b\|_{C^0([0,T])}\right)\le C\|\bar\Phi-\Phi\|_{C^0([0,T])}.$$

Here and hereafter, we denote by C various constants that are independent of $(\bar\Phi,\Phi)$. Therefore,

$$\sup_{t\in[0,t_0^{a,b}]}|(\mathcal{F}_2(\bar\Phi)-\mathcal{F}_2(\Phi))(t)|\le C\nu_1\|\bar\Phi-\Phi\|_{C^0([0,T])}. \tag{6.19}$$

For any given $t\in[t_0^{\bar a,\bar b},T]$, by (6.15) and (6.18), we obtain

$$|(\mathcal{F}_2(\bar\Phi)-\mathcal{F}_2(\Phi))(t)|=|h(\tau^{\bar a,\bar b}(t))-h(\tau^{a,b}(t))|\le\nu_1|\tau^{\bar a,\bar b}(t)-\tau^{a,b}(t)|. \tag{6.20}$$

By (6.9), for every $t\in[t_0^{a,b},T]$,

$$\xi^{a,b}(s;t,1)=e^{\int_s^t\frac{l^1-l^0}{Ta(\sigma)}\,d\sigma}-\int_s^t\frac{\zeta N^{a,b}(\sigma)}{a(\sigma)}\cdot e^{\int_s^\sigma\frac{l^1-l^0}{Ta(s)}\,ds}\,d\sigma,\quad s\in[\tau^{a,b}(t),T].$$

Therefore,

$$\sup_{s\in[\tau^{\bar{a},\bar{b}}(t),T]} |\xi^{\bar{a},\bar{b}}(s;t,1)-\xi^{a,b}(s;t,1)| \le C\left(\|\bar{a}-a\|_{C^0[0,T]}+\|\bar{b}-b\|_{C^0[0,T]}\right). \quad (6.21)$$

Then, for any $t\in[t_0^{\bar{a},\bar{b}},T]$, we have

$$\begin{aligned}|\tau^{\bar{a},\bar{b}}(t)-\tau^{a,b}(t)| \le&\frac{1}{\inf\alpha_p^{a,b}}\left|\int_{\tau^{a,b}(t)}^{\tau^{\bar{a},\bar{b}}(t)}\alpha_p^{a,b}(s;\xi^{a,b}(s;t,1))\,ds\right|\\ =&\frac{1}{\inf\alpha_p^{a,b}}\left|\int_{\tau^{\bar{a},\bar{b}}(t)}^{T}\left(\alpha_p^{a,b}(s;\xi^{a,b}(s;t,1))-\alpha_p^{\bar{a},\bar{b}}(s;\xi^{\bar{a},\bar{b}}(s;t,1))\right)\,ds\right|\\ \le&C\left(\|\bar{a}-a\|_{C^0([0,T])}+\|\bar{b}-b\|_{C^0([0,T])}\right)\\ \le&C\|\bar{\Phi}-\Phi\|_{C^0([0,T])},\end{aligned} \quad (6.22)$$

where

$$\inf\alpha_p^{a,b}:=\inf_{\substack{(t,x)\in[0,T]\times[0,1]\\(a,b)\in\Omega^{\varepsilon_1,T}}}\alpha_p^{a,b}(t,x)>0$$

is independent of Φ. Therefore,

$$\sup_{t\in\left[t_0^{\bar{a},\bar{b}},T\right]}|(\mathcal{F}_2(\bar{\Phi})-\mathcal{F}_2(\Phi))(t)|\le C\nu_1\|\bar{\Phi}-\Phi\|_{C^0([0,T])}. \quad (6.23)$$

For any given $t\in\left[t_0^{a,b},t_0^{\bar{a},\bar{b}}\right]$, by (6.16) and (6.18), we have

$$\begin{aligned}|\mathcal{F}_2(\bar{\Phi})-\mathcal{F}_2(\Phi)| =&|f_p^0(\beta^{\bar{a},\bar{b}}(t))-h(\tau^{a,b}(t))|\\ \le&|f_p^0(\beta^{\bar{a},\bar{b}}(t))-f_p^0(0)|+|h(\tau^{a,b}(t))-h(0)|\\ \le&\|f_{p_x}^0\|_{L^\infty}|\beta^{\bar{a},\bar{b}}(t)|+\|h_t\|_{L^\infty}|\tau^{a,b}(t)|\\ \le&\nu_1(|\beta^{\bar{a},\bar{b}}(t)|+|\tau^{a,b}(t)|).\end{aligned} \quad (6.24)$$

Note that for any $t\in\left[t_0^{a,b},t_0^{\bar{a},\bar{b}}\right]$,

$$\xi^{\bar{a},\bar{b}}(s;t,1)=e^{\int_s^t\frac{J^1-J^0}{T\bar{a}(\sigma)}\,d\sigma}-\int_s^t\frac{\zeta N^{\bar{a},\bar{b}}(\sigma)}{\bar{a}(\sigma)}\cdot e^{\int_s^\sigma\frac{J^1-J^0}{T\bar{a}(s)}\,ds}\,d\sigma,\quad s\in[0,t].$$

Then, using $\xi^{\bar{a},\bar{b}}(0;t_0^{\bar{a},\bar{b}},1)=0$, it follows that

$$|\beta^{\bar{a},\bar{b}}(t)|=|\xi^{\bar{a},\bar{b}}(0;t,1)-\xi^{\bar{a},\bar{b}}\left(0;t_0^{\bar{a},\bar{b}},1\right)|\le C|t-t_0^{\bar{a},\bar{b}}|\le C|t_0^{\bar{a},\bar{b}}-t_0^{a,b}|. \quad (6.25)$$

On the other hand, by (6.12) and note that $\tau^{a,b}(t_0^{a,b})=0$, we can prove that for any $t\in\left[t_0^{a,b},t_0^{\bar{a},\bar{b}}\right]$,

$$|\tau^{a,b}(t)|=|\tau^{a,b}(t)-\tau^{a,b}\left(t_0^{a,b}\right)|\le C|t-t_0^{a,b}|\le C|t_0^{\bar{a},\bar{b}}-t_0^{a,b}|. \quad (6.26)$$

By definition of $t_0^{\bar{a},\bar{b}}$ and $t_0^{a,b}$, similarly to the derivation of (6.21) and (6.22), we obtain

$$|t_0^{\bar{a},\bar{b}}-t_0^{a,b}|\le C\left(\|\bar{a}-a\|_{C^0([0,T])}+\|\bar{b}-b\|_{C^0([0,T])}\right)\le C\|\bar{\Phi}-\Phi\|_{C^0([0,T])}. \quad (6.27)$$

Then it follows from (6.24)–(6.27) that

$$\sup_{t\in\left[t_0^{a,b},t_0^{\bar{a},\bar{b}}\right]}|(\mathcal{F}_2(\bar{\Phi})-\mathcal{F}_2(\Phi))(t)|\le C\nu_1\|\bar{\Phi}-\Phi\|_{C^0([0,T])}. \quad (6.28)$$

Combining (6.19), (6.23), and (6.28), we can choose ν_1 small enough such that

$$\|\mathcal{F}(\bar{\Phi})-\mathcal{F}(\Phi)\|_{C^0([0,T])}=\|\mathcal{F}_2(\bar{\Phi})-\mathcal{F}_2(\Phi)\|_{C^0([0,T])}\le\frac{1}{2}\|\bar{\Phi}-\Phi\|_{C^0([0,T])}. \quad (6.29)$$

Then the contraction mapping theorem implies that $\mathcal{F}$ has a unique fixed point $(l(\cdot),f_p(\cdot,1))$ in $\Omega^{\varepsilon_1,T}$. Therefore, we find a solution (l,f_p) to Cauchy problems (2.2)–(2.4), which also satisfies the final conditions (3.9) and (3.10). Moreover, because of the uniqueness of the solution, the desired control functions $N(t)$ and $F_{in}(t)$ can be chosen by substituting the fixed point $(l(\cdot),f_p(\cdot,1))$ into (6.5) and (6.14), respectively. This concludes the proof of Theorem 3.3.

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