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Adaptive boundary control of reaction–diffusion PDEs with unknown input delay[☆]

Shanshan Wang [a,b], Jie Qi [a,c], Mamadou Diagne [d,*]

[a] College of Information science and Technology, Donghua University, Shanghai, 201620, China
[b] Engineering research Center of Digitized Textile and Fashion Technology of Ministry Education, Donghua University, Shanghai, 201620, China
[c] State Key Laboratory of Synthetical Automation for Process Industries, Shenyang, Liaoning, 110819, China
[d] Department of Mechanical Aerospace and Nuclear Engineering, Rensselaer Polytechnic Institute, Troy, NY, 12180, United States of America

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ABSTRACT

We design an adaptive full-state feedback controller to stabilize a one-dimensional reaction–diffusion equation with unknown boundary input delay. An infinite-dimensional representation of the actuator delay is utilized to transform the system into a transport PDE cascading with a reaction–diffusion PDE. A suitable parameter update law is designed to establish local boundedness of the system trajectories and asymptotic convergence stability result using the well-known PDE backstepping technique and a Lyapunov argument. Consistent simulation results are provided to support the theoretical results.

1. Introduction

The control of diffusion PDEs (partial differential equations) still stirs up a lot of interest owing to their impact on innumerable physical systems. Various classes of spatio-temporal diffusion PDEs have been used to characterize the dynamics of relevant engineering processes including direct-contact membrane distillation processes (Eleiwi & Laleg-Kirati, 2018), tubular reactor (Boskovic & Krstic, 2002; Orlov & Dochain, 2002), temperature regulation over a catalytic bar (Dubljevic, Kobilarov, & Ng, 2010), information spreading in online social media (Lei, Lin, & Wang, 2013), Lithium-ion batteries (Forman, Bashash, Stein, & Fathy, 2011) and multi-agent systems (Ferrari-Trecate, Buffa, & Gati, 2006; Qi, Tang and Wang, 2019).

The literature devoted to the boundary control of parabolic PDEs has flourished substantially within the past two decades. Early contributions employed power series solutions to design flatness-based boundary feedback controllers for linear and nonlinear diffusion systems (Meurer, Becker, & Zeitz, 2003; Meurer & Zeitz, 2003), which have been later extended to systems with higher-dimensional spatial domains (Meurer & Kugi, 2009). Motivated by the stabilization of unstable heat equations, an alternative control approach known as PDE backstepping technique emerges as a systematic design methodology (Boskovic, Krstic, & Liu, 2001; Krstic & Smyshlyaev, 2008). Backstepping employs a change of coordinate, namely, a Volterra transformation that maps originally unstable PDEs into a stable target system to enable direct deduction of stabilizing boundary controllers from the invertibility of the transformation. Over the last few years, the aforesaid method has been successfully expanded to exponentially stabilize challenging diffusion processes involving the 3-D formation of multi-agents (Qi, Vazquez, & Krstic, 2015), coupled reaction–diffusion systems with the same diffusivity (Baccoli, Orlov, & Pisano, 2014) or spatially distributed coefficients (Vazquez & Krstic, 2017) and several linear and nonlinear ordinary differential equations having a diffusion PDE actuator dynamics in fixed or time-varying spatial domains (Koga, Diagne, & Krstic, 2019; Krstic, 2009a; Tang & Xie, 2011). Further studies investigated the control of scalar reaction–diffusion PDEs subject to constant boundary input delays considering a dead time arising from the physical constraints such as communication lag times. In this case, the representation of the delay

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* Corresponding author.
E-mail addresses: wss_dhu@126.com (S. Wang), jieqi@dhu.edu.cn (J. Qi), diagnm@rpi.edu (M. Diagne).

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by a transport PDE cascading with the plant reaction–diffusion dynamics has been exploited to compensate for the effect of the known but arbitrarily large delay with the help of backstepping design (Krstic, 2009b). Along the same line, our recent results applied the backstepping technique to design a delay-compensated boundary controller for multi-agent systems described by a 3-D reaction–diffusion equation defined on a cylindrical topology when the delay acting at the boundary input is known (Qi, Wang, Fang and Diagne, 2019; Wang, Qi, & Fang, 2017). As well, backstepping method has been employed to compensate spatially input distributed delays in reaction–diffusion PDEs (Qi & Krstic, 2021). Alternatively, a new design methodology has led to the construction of a boundary controller based on an explicit form of the classical Artstein transformation for the finite-dimensional unstable part of the delay system derived from the expansion of a 1-D reaction–diffusion PDE solutions as series of basis eigenfunctions (Prieur & Trelat, 2019). The same method has been employed to stabilize several PDEs with input delays including the linear Kuramoto–Sivashinsky equation (Guzmán, Marx, & Cerpa, 2019), diagonal infinite-dimensional systems (Lhachemi & Prieur, 2021) with the analysis of robustness with respect to input delays (Lhachemi, Prieur, & Shorten, 2019).

On the other hand, in Smyshlyaev and Krstic (2010), the design of adaptive backstepping boundary controllers has been developed for various class of diffusion PDEs with unknown destabilizing parameters such as the diffusivity, the reaction, or even boundary coefficients. For hyperbolic PDEs, we refer the reader to the rich literature (Anfinsen & Aamo, 2017a, 2017b, 2017c, 2018a, 2018b; Anfinsen, Diagne, Aamo, & Krstic, 2016, 2017; Bernard & Krstic, 2014). The aforementioned results are built extending the three-parameter identifiers proposed for nonlinear ODEs (Krstic, Kanellakopoulos, & Kokotovic, 1995), enabling the boundary adaptive control of several delay-free 1-D PDEs based on Lyapunov, passivity, or swapping approaches. We emphasize that the Lyapunov approach is known to provide superior transient performance properties as stated in Krstic and Smyshlyaev (2008). Also, the sliding mode approach employed in Orlov, Fradkov, and Andrievsky (2020) provides good performance to adaptively stabilize uncertain distributed parameters as well as the model reference adaptive control techniques (Bentsman & Orlov, 2001; Demetriou & Rosen, 1994; Smyshlyaev, Orlov, & Krstic, 2010).

In our present contribution, we consider the problem of stabilizing a reaction–diffusion system with unknown and arbitrarily large boundary input delay. The adaptive Lyapunov design used in Bresch-Pietri and Krstic (2010) and Krstic and Bresch-Pietri (2009) to construct delay-adaptive predictor feedback controllers for linear ODE systems with unknown input delay is employed to develop a delay-adaptive predictor feedback boundary controller for the considered system. *Possibly, our result represents the first delay-adaptive PDE control scheme ever developed.* Our design relies on the backstepping method and our choice of the unknown delay parameter's update law leads to a target system structured as a mixed PDE–PDE cascade system. Using a Lyapunov argument, we establish the local boundedness of the system trajectory and an asymptotic convergence stability result. The invertibility of the backstepping transformation enables to state the norm equivalence between the target system and the plant dynamics resulting in local stability of the adaptively controlled plant. Compared to Krstic and Bresch-Pietri (2009), the system structure leads to an unbounded input operator at the boundary of the PDE–PDE cascade system and only local stability result in H^1 norm of the actuator state can be achieved via a Lyapunov-based adaptive control design. It is worth mentioning that backstepping has been used to stabilize both nonlinear ODEs with complex input delays (Diagne, Bekiaris-Liberis and Krstic, 2017; Diagne, Bekiaris-Liberis, Otto and Krstic, 2017) or uncertain linear time-delay systems (Zhu & Krstic, 2015; Zhu, Krstic, & Su, 2017). For identifying parameters and delays in linear ODEs with measurable states, a different approach exploiting the weak controllability argument has been developed in Belkoura and Orlov (2002).

The paper is organized as follows. In Section 2 we briefly recall the non-adaptive controller for the considered class of system. Section 3 discusses the adaptive controller design in the presence of an unknown boundary input delay. In Section 4, the closed-loop system's stability is stated and consistent simulation results are shown in Section 5. The paper ends with concluding remarks in Section 6.

Notation: Throughout the paper, we defined the L^2 norm for a function $\chi(x) \in L^2[0,1]$ as $\|\chi\|^2 = \|\chi\|^2_{L^2} = \int_0^1 |\chi(x)|^2 dx$, and the H^1 norm for a function $\chi(x) \in L^2[0,1]$ as $\|\chi\|^2_{H^1} = \|\chi\|^2_{L^2} + \|\chi_x\|^2_{L^2}$.

We introduce the Bessel function J_n and the modified Bessel function I_n, where $n = \{1, 2, 3, \ldots, 5\}$.

For any given function $\psi(\cdot, \hat{D}(t))$, the following holds:

$$\frac{\partial \psi(\cdot, \hat{D}(t))}{\partial t} = \dot{\hat{D}}(t) \frac{\partial \psi(\cdot, \hat{D}(t))}{\partial \hat{D}(t)}. \tag{1}$$

2. Problem formulation and non-adaptive boundary controller

Consider the scalar reaction–diffusion PDE with a known and arbitrarily long actuator delay $D > 0$ defined as

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{2}$$

$$u(0,t) = 0, \tag{3}$$

$$u(1,t) = U(t-D), \tag{4}$$

where the full-state $u(x,t)$, $(x,t) \in (0,1) \times \mathbb{R}_+$ is measurable and the constant parameter λ is known. As established in Krstic (2009b), the delayed input $U(t-D)$ can be written as a transport equation cascading into (2) considering the infinite-dimensional actuator state $v(x,t) = U(t + D(x-1))$. Thus, system (2)–(4) is equivalent to

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{5}$$

$$u(0,t) = 0, \quad u(1,t) = v(0,t), \tag{6}$$

$$D v_t(x,t) = v_x(x,t), \tag{7}$$

$$v(1,t) = U(t). \tag{8}$$

Following Krstic (2009b), an exponentially stabilizing boundary controller for the equivalent cascade system (5)–(8) is defined as follows

$$U(t) = \int_0^1 \gamma(1,y) u(y,t) dy + D \int_0^1 q(1,y) v(y,t) dy, \tag{9}$$

where $\gamma(x,y)$ and $q(x,y)$ are the controller gain kernels, which satisfy the following equations:

$$\gamma_x(x,y) = D\gamma_{yy}(x,y) + D\lambda\gamma(x,y), \quad (x,y) \in (0,1], \tag{10}$$

$$\gamma(x,1) = \gamma(x,0) = 0, \tag{11}$$

$$\gamma(0,y) = k(1,y), \tag{12}$$

$$q_x(x,y) = -q_y(x,y), \quad (x,y) \in [0,1], \tag{13}$$

$$q(x,0) = -\gamma_y(x,1), \tag{14}$$

where the kernel $k(x,y)$ is governed by the following well-posed PDE:

$$k_{xx}(x,y) = k_{yy}(x,y) + \lambda k(x,y), \quad (x,y) \in [0,1], \tag{15}$$

$$k(x,0)=0,\quad k(x,x)=-\frac{\lambda}{2}x. \tag{16}$$

The solution of the governing equations of $k(x,y)$, $q(x,y)$ and $\gamma(x,y)$ are explicitly given as

$$k(x,y)=-\lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \tag{17}$$

$$q(x,y)=-\gamma_y(x-y,1), \tag{18}$$

$$\gamma(x,y)=2\sum_{n=1}^{\infty}e^{D(\lambda-n^2\pi^2)x}\sin(n\pi y)\cdot\int_0^1\sin(n\pi s)k(1,s)\mathrm{d}s, \tag{19}$$

where I_1 is an appropriate modified Bessel function and $x\in[0,1]$.

The control law $U(t)$ defined in (9) is a delay-compensated feedback controller whose second term cancels the destabilizing effect of the actuator dynamics. The H^1 exponential stability of the closed-loop system consisting of (5), (7) together with the controller (9) can be derived employing the following backstepping transformation

$$w(x,t)=u(x,t)-\int_0^x k(x,y)u(y,t)\mathrm{d}y, \tag{20}$$

$$z(x,t)=v(x,t)-\int_0^1\gamma(x,y)u(y,t)\mathrm{d}y - D\int_0^x q(x,y)v(y,t)\mathrm{d}y, \tag{21}$$

together with the stable target system

$$w_t(x,t)=w_{xx}(x,t), \tag{22}$$

$$w(0,t)=0, \tag{23}$$

$$w(1,t)=z(0,t), \tag{24}$$

$$Dz_t(x,t)=z_x(x,t), \tag{25}$$

$$z(1,t)=0. \tag{26}$$

The inverse of the transformation (20), (21) is given by

$$u(x,t)=w(x,t)+\int_0^x l(x,y)w(y,t)\mathrm{d}y, \tag{27}$$

$$v(x,t)=z(x,t)+\int_0^1\eta(x,y)w(y,t)\mathrm{d}y + D\int_0^x p(x,y)z(y,t)\mathrm{d}y, \tag{28}$$

where gain kernels $l(x,y)$, $\eta(x,y)$ and $p(x,y)$ satisfy the following partial differential equations

$$l_{xx}(x,y)=l_{yy}(x,y)-\lambda l(x,y),\quad (x,y)\in[0,1], \tag{29}$$

$$l(x,0)=0, \tag{30}$$

$$l(x,x)=-\frac{\lambda}{2}x, \tag{31}$$

$$\eta_x(x,y)=D\eta_{yy}(x,y),\quad (x,y)\in(0,1], \tag{32}$$

$$\eta(x,1)=\eta(x,0)=0, \tag{33}$$

$$\eta(0,y)=k(1,y), \tag{34}$$

$$p_x(x,y)=-p_y(x,y),\quad (x,y)\in[0,1], \tag{35}$$

$$p(x,0)=-\eta_y(x,1). \tag{36}$$

The solution to the equations of $l(x,y)$, $p(x,y)$ and $\eta(x,y)$ are computed as

$$l(x,y)=-\lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}, \tag{37}$$

$$p(x,y)=-\eta_y(x-y,1), \tag{38}$$

$$\eta(x,y)=2\sum_{n=1}^{\infty}e^{-Dn^2\pi^2x}\sin(n\pi y)\cdot\int_0^1\sin(n\pi s)l(1,s)\mathrm{d}s, \tag{39}$$

where J_1 is an appropriate Bessel function and $x\in[0,1]$.

The derivation of such backstepping controllers has been developed in many contributions. Readers are referred to Krstic (2009b), Qi, Wang et al. (2019) and Wang et al. (2017) where the design of the controller is extensively discussed.

3. Design of a delay-adaptive boundary feedback control law

3.1. Adaptive controller design

Considering the plant (2)–(4) with an unknown delay D or equivalently the cascade system (5)–(8) with an unknown spatial domain length D, our goal is to design an adaptive boundary controller that stabilizes the system's dynamics under the following assumption.

Assumption 1. The upper and lower bounds of $D>0$, denoted $\bar{D}$ and $\underline{D}$, respectively, are known.

Based on the certainty equivalence principle, we define the following adaptive controller

$$U(t)=\int_0^1\gamma(1,y,\hat{D}(t))u(y,t)\mathrm{d}y + \hat{D}(t)\int_0^1 q(1,y,\hat{D}(t))v(y,t)\mathrm{d}y, \tag{40}$$

which is similar to (9), but accounts for the estimate of D defined as $\hat{D}(t)$. The estimate $\hat{D}(t)$ is governed by the update law $\dot{\hat{D}}(t)$, which is given in Section 3.3.

3.2. Target system for the plant with unknown input delay

To prove the stability of the plant (2)–(4), equivalently, the system (5)–(8) under the control law (40), we introduce the volterra transformation $(u,v)\mapsto(w,z)$ consisting of (20) and

$$z(x,t)=v(x,t)-\int_0^1\gamma(x,y,\hat{D}(t))u(y,t)\mathrm{d}y - \hat{D}(t)\int_0^x q(x,y,\hat{D}(t))v(y,t)\mathrm{d}y, \tag{41}$$

whose inverse is defined as (27) together with

$$v(x,t)=z(x,t)+\int_0^1\eta(x,y,\hat{D}(t))w(y,t)\mathrm{d}y + \hat{D}(t)\int_0^x p(x,y,\hat{D}(t))z(y,t)\mathrm{d}y. \tag{42}$$

The gain kernels of the transformations (41) and (42) depend on the estimated value of the unknown delay $\hat{D}(t)$ and are consequently time-dependent whereas (21) and (28) are static kernel functions involving a known and constant boundary input delay. Using (20) and (41), system (5)–(8) maps into the following target system

$$w_t(x,t)=w_{xx}(x,t), \tag{43}$$

$$w(0,t)=0, \tag{44}$$

$$w(1,t)=z(0,t), \tag{45}$$

$$Dz_t(x,t)=z_x(x,t)-\tilde{D}(t)P_1(x,t)-D\dot{\hat{D}}(t)P_2(x,t), \tag{46}$$

$$z(1,t)=0, \tag{47}$$

where $\tilde{D}(t) = D - \hat{D}(t)$ is the estimation error, $P_i(x,t),\ i = \{1,2\}$ are functions defined below:

$$P_1(x,t) = M_1(x, D(t))z(0,t) + \int_0^1 w(y,t)M_2(x,y,\hat{D}(t))\mathrm{d}y, \tag{48}$$

$$P_2(x,t) = \int_0^x z(y,t)M_3(x,y,\hat{D}(t))\mathrm{d}y + \int_0^1 w(y,t)M_4(x,y,\hat{D}(t))\mathrm{d}y, \tag{49}$$

where $M_i,\ i = \{1,2,3,4\}$, are the following functions

$$M_1(x,\hat{D}(t)) = -\gamma_y(x,1,\hat{D}(t)), \tag{50}$$

$$M_2(x,y,\hat{D}(t)) = -\gamma_y(x,1,\hat{D}(t))l(1,y) + \frac{1}{\hat{D}(t)}\gamma_x(x,y,\hat{D}(t)) + \frac{1}{\hat{D}(t)}\int_y^1 \gamma_x(x,\xi,\hat{D}(t))l(\xi,y)\mathrm{d}\xi, \tag{51}$$

$$\begin{aligned} M_3(x,y,\hat{D}(t)) =& \hat{D}(t)^2\int_y^x q_{\hat{D}(t)}(x,\xi,\hat{D}(t))p(\xi,y,\hat{D}(t))\mathrm{d}\xi \\ &+ \hat{D}(t)\int_y^x q(x,\xi,\hat{D}(t))p(\xi,y,\hat{D}(t))\mathrm{d}\xi \\ &+ q(x,y,\hat{D}(t)) + \hat{D}(t)q_{\hat{D}(t)}(x,y,\hat{D}(t)), \end{aligned} \tag{52}$$

$$\begin{aligned} M_4(x,y,\hat{D}(t)) =& \hat{D}(t)\int_0^x q_{\hat{D}(t)}(x,\xi,\hat{D}(t))\eta(\xi,y,\hat{D}(t))\mathrm{d}\xi \\ &+ \int_0^x q(x,\xi,\hat{D}(t))\eta(\xi,y,\hat{D}(t))\mathrm{d}\xi \\ &+ \int_y^1 \gamma_{\hat{D}(t)}(x,\xi,\hat{D}(t))l(\xi,y)\mathrm{d}\xi \\ &+ \gamma_{\hat{D}(t)}(x,y,\hat{D}(t)). \end{aligned} \tag{53}$$

We introduce the following transformation to create homogeneous boundary conditions for the target system (43)–(47)

$$\check{w}(x,t) = w(x,t) - xz(0,t), \tag{54}$$

which leads to the following PDE cascade system:

$$\check{w}_t(x,t) = \check{w}_{xx}(x,t) - xz_t(0,t), \tag{55}$$

$$\check{w}(0,t) = \check{w}(1,t) = 0, \tag{56}$$

$$Dz_t(x,t) = z_x(x,t) - \tilde{D}(t)\check{P}_1(x,t) - D\dot{\hat{D}}(t)\check{P}_2(x,t), \tag{57}$$

$$z(1,t) = 0, \tag{58}$$

where

$$\check{P}_1(x,t) = M_1(x,\hat{D}(t))z(0,t) + \int_0^1 \check{w}(y,t)M_2(x,y,\hat{D}(t))\mathrm{d}y + \int_0^1 yz(0,t)M_2(x,y,\hat{D}(t))\mathrm{d}y, \tag{59}$$

$$\check{P}_2(x,t) = \int_0^x z(y,t)M_3(x,y,\hat{D}(t))\mathrm{d}y + \int_0^1 (\check{w}(y,t) + yz(0,t))M_4(x,y,\hat{D}(t))\mathrm{d}y, \tag{60}$$

3.3. The parameter's update law

To estimate the unknown parameter D, we choose the following update law

$$\dot{\hat{D}}(t) = \theta \mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\}, \quad 0 < \theta < 1, \tag{61}$$

where $\tau(t)$ is given as

$$\tau(t) = -2\int_0^1 (1+x)z(x,t)\check{P}_1(x,t)\mathrm{d}x, \tag{62}$$

and the standard projection operator is defined as follows

$$\mathrm{Proj}_{[\underline{D},\bar{D}]}\{\tau(t)\} = \begin{cases} 0 & \hat{D}(t) = \underline{D} \text{ and } \tau(t) < 0, \\ 0 & \hat{D}(t) = \bar{D} \text{ and } \tau(t) > 0, \\ \tau(t) & \text{otherwise.} \end{cases} \tag{63}$$

Remark 1. The projection is needed to keep the parameter $\hat{D}(t)$ within a priori set and cannot be viewed as a robustification tool (Krstic & Smyshlyaev, 2008). In this case, it enables fast adaption and ensures that the adaptive parameter $\hat{D}(t)$ does not exceed its known maximum value $\bar{D}$ or fall below its known minimum value $\underline{D}$. In that sense, it prevents adaption transients by restricting the size of the adaption gain.

The local stability result of the closed-loop system consisting of (5)–(8), with update law (61)–(63) and the adaptive controller (40), is stated in the following Theorem.

Theorem 1. *Consider the closed-loop system consisting of the plant (5)–(8), the control law (40), the update law (61)–(63) under Assumption 1. Assuming the well-posedness of the closed-loop system, local boundedness and asymptotic convergence of the system trajectories are guaranteed and there exist positive constants ρ, R such that if the initial conditions $(u_0, v_0, \hat{D}_0)$ satisfy $\Psi(0) < \rho$, where*

$$\Psi(t) = \int_0^1 u(x,t)^2\mathrm{d}x + \int_0^1 v(x,t)^2\mathrm{d}x + \int_0^1 v_x(x,t)^2\mathrm{d}x + v(0,t)^2 + \tilde{D}(t)^2, \tag{64}$$

the following holds:

$$\Psi(t) \le R\Psi(0), \quad \forall t \ge 0. \tag{65}$$

Furthermore,

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| = 0, \tag{66}$$

$$\lim_{t\to\infty}\max_{x\in[0,1]}|v(x,t)| = 0. \tag{67}$$

Remark 2. We emphasize that only a local stability result can be ensured in our case due to the following technical issue. In fact, the PDE–PDE cascade system (43)–(47) connected through the boundary generates an unbounded input operator, which requires the H^1 norm of the actuator state to establish the stability proof with the help of a Lyapunov argument. Taking the derivative of the Lyapunov function, the appearance of terms involving $z_x(1,t) \neq 0$ in the stability proof prevents the statement of a global stability result. Similarly, for the ODE case in Krstic and Bresch-Pietri (2009), the global stability result cannot be obtained considering the H^1 norm of the actuator state in the Lyapunov function.

4. Proof of the local stability of the delay-adaptive closed-loop system

The local stability of the (u, v)-system (5)–(8) under the control law (40), and the update law (61)–(63) is established with the following steps:

- First, we establish the norm equivalence stated in Proposition 1, which will be used to deduce the stability of the (u, v)-system knowing that of the $(\check{w}, z)$-system (55)–(58).
- Second, we construct a Lyapunov function that ensures the local stability of the $(\check{w}, z)$-system.
- Then, we establish the regulation of the state $u(x,t)$ and $v(x,t)$.

Proposition 1. *The following estimates hold between the state of the original system (5)–(8), and the state of the target system (55)–(58):*

$$\|u\|^2+\|v\|^2+\|v_x\|^2+v(0,t)^2 \le r_1\|\check{w}\|^2+r_2\|z\|^2+r_3\|z_x\|^2+r_4z(0,t)^2, \tag{68}$$

$$\|\check{w}\|^2+\|z\|^2+\|z_x\|^2+z(0,t)^2 \le s_1\|u\|^2+s_2\|v\|^2+s_3\|v_x\|^2+s_4v(0,t)^2, \tag{69}$$

where r_i and s_i, $i=\{1,2,3,4\}$, are sufficiently large positive constants given by

$$r_1 \ge 4\left(1+\int_0^1\int_0^x l(x,y)^2dydx\right)+3\int_0^1 l(1,y)^2dy + \frac{8}{\underline{D}\pi^2}\int_0^1 l(1,y)^2dy+\frac{10\bar{D}}{\pi^2}\int_0^1 l_{yy}(1,y)^2dy, \tag{70}$$

$$r_2 \ge 4+5\bar{D}^2l_y(1,1)^2+\frac{\bar{D}^4}{2}\int_0^1\left(5l_{yyy}(1,y)^2+\frac{4}{\underline{D}^2}l_y(1,y)^2\right)dy + \frac{\bar{D}^2(\bar{D}+1)}{2}\int_0^1(5l_{yy}(1,y)^2+\frac{4}{\underline{D}^2}l(1,y)^2)dy, \tag{71}$$

$$r_3 \ge 5, \tag{72}$$

$$r_4 \ge 3+\frac{4}{3}\left(1+\int_0^1\int_0^x l(x,y)^2dydx\right)+\int_0^1 l(1,y)^2dy + \frac{8}{3\underline{D}\pi^2}\int_0^1 l(1,y)^2dy+\frac{10\bar{D}}{3\pi^2}\int_0^1 l_{yy}(1,y)^2dy, \tag{73}$$

$$s_1 \ge 4+4\int_0^1\int_0^x k(x,y)^2dydx+16\beta\bar{D}\int_0^1 k_{yy}(1,y)^2dy + \left(\frac{10}{3}+\frac{6\beta}{\underline{D}}+16\beta\bar{D}\lambda^2\right)\int_0^1 k(1,y)^2dy, \tag{74}$$

$$s_2 \ge 3+\bar{D}^2\left(2\lambda\beta\left(4\bar{D}^2\lambda+2\bar{D}+1\right)\left(\frac{3}{\underline{D}^2}+16\lambda^2\right) + \frac{3(\bar{D}+1)}{2\underline{D}^2}\right)\int_0^1 k(1,y)^2dy+\bar{D}^2(4\lambda(4\bar{D}^2\lambda+2\bar{D}+1) + 3)\int_0^1\left(k_y(1,y)^2+\lambda k(1,y)^2\right)dy+4\bar{D}^2k_y(1,1)^2 + 4\bar{D}^2(2\bar{D}^2\lambda+\bar{D}+1)\int_0^1\left(k_{yy}(1,y)^2+\lambda^2k(1,y)^2\right)dy + 8\bar{D}^4\int_0^1\left(k_{yyy}(1,y)^2+\lambda^2k_y(1,y)^2\right)dy, \tag{75}$$

$$s_3 \ge 4, \quad s_4 \ge \frac{10}{3}, \tag{76}$$

where $\beta=\frac{e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}}{4\lambda-\pi^2}$.

The proof of Proposition 1 is stated in Appendix B.

Next, we show the local stability for the closed-loop system consisting of the (u,v)-system under the control law (40), and with the update law (61)–(63).

4.1. Local stability of the closed-loop system

To establish the local stability of the target system (55)–(58), we introduce a Lyapunov-Krasovskii-type function

$$V_1(t)=AD\int_0^1\check{w}(x,t)^2dx+D\int_0^1(1+x)(z(x,t)^2 + z_x(x,t)^2)dx+\frac{D}{2}z(0,t)^2+\frac{\tilde{D}(t)^2}{2\theta}, \tag{77}$$

where A is a positive constant.

Since our Lyapunov function involves the H^1 norm of $z(x,t)$, we define the $z_x(x,t)$-system, by taking the partial derivative of (57) with respect to x combined with the partial derivative of (58) with respect to t. The following PDE is obtained:

$$Dz_{xt}(x,t)=z_{xx}(x,t)-\tilde{D}(t)\check{P}_{1x}(x,t)-D\dot{\hat{D}}(t)\check{P}_{2x}(x,t), \tag{78}$$

$$z_x(1,t)=\tilde{D}(t)\check{P}_1(1,t)+D\dot{\hat{D}}(t)\check{P}_2(1,t), \tag{79}$$

where $\check{P}_{1x}(x,t)$ and $\check{P}_{2x}(x,t)$ are the partial derivatives of $\check{P}_1(x,t)$ and $\check{P}_2(x,t)$ with respect to x, respectively.

Taking the time derivative of (77) along (55)–(58), (78) and (79), we get

$$\begin{aligned}\dot{V}_1(t)=&2AD\int_0^1\check{w}(x,t)\check{w}_{xx}(x,t)dx+AE_1(t)\\&+2\int_0^1(1+x)z(x,t)z_x(x,t)dx+z(0,t)z_x(0,t)\\&+2\int_0^1(1+x)z_x(x,t)z_{xx}(x,t)dx\\&+\tilde{D}(t)\left(2A\int_0^1\check{w}(x,t)x\check{P}_1(0,t)dx-z(0,t)\check{P}_1(0,t)\right.\\&-2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)dx\\&\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)dx\right)-\tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\&+D\dot{\hat{D}}(t)\left(2A\int_0^1\check{w}(x,t)x\check{P}_2(0,t)dx-z(0,t)\check{P}_2(0,t)\right.\\&-2\int_0^1(1+x)z(x,t)\check{P}_2(x,t)dx\\&\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{2x}(x,t)dx\right),\end{aligned} \tag{80}$$

where

$$E_1(t)=-2\int_0^1\check{w}(x,t)xz_x(0,t)dx. \tag{81}$$

Using integration by parts, Cauchy–Schwarz and Young's inequalities, we derive the following estimate

$$\begin{aligned}E_1(t)=&\int_0^1\check{w}_x(x,t)x^2z_x(0,t)dx,\\ \le&\left(\int_0^1\check{w}_x(x,t)^2dx\right)^{\frac{1}{2}}\left(\int_0^1x^4z_x(0,t)^2dx\right)^{\frac{1}{2}},\\ \le&\frac{1}{2\iota_1}\|w_x\|^2dx+\frac{\iota_1}{10}z_x(0,t)^2.\end{aligned} \tag{82}$$

Thus,

$$\begin{aligned}\dot{V}_1(t)\le&-2AD\|\check{w}_x\|^2+\frac{A}{2\iota_1}\|\check{w}_x\|^2+\frac{A\iota_1}{10}z_x(0,t)^2-z(0,t)^2\\&-\|z\|^2-\|z_x\|^2+2z_x(1,t)^2-z_x(0,t)^2+\frac{1}{2\iota_2}z(0,t)^2\\&+\frac{\iota_2}{2}z_x(0,t)^2+\tilde{D}(t)\left(2A\int_0^1\check{w}(x,t)x\check{P}_1(0,t)dx\right.\\&-z(0,t)\check{P}_1(0,t)-2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)dx\\&\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)dx\right)-\tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\&+D\dot{\hat{D}}(t)\left(2A\int_0^1\check{w}(x,t)x\check{P}_2(0,t)dx-z(0,t)\check{P}_2(0,t)\right.\end{aligned}$$

$$- 2\int_0^1 (1+x)z(x,t)\check{P}_2(x,t)\mathrm{d}x$$
$$-2\int_0^1 (1+x)z_x(x,t)\check{P}_{2x}(x,t)\mathrm{d}x\Bigg). \quad (83)$$

From (79), we obtain the following relation

$$\begin{aligned}2z_x^2(1,t) =& 2(\tilde{D}(t)\check{P}_1(1,t)+D\dot{\hat{D}}(t)\check{P}_2(1,t))^2,\\ \le& 4\tilde{D}(t)^2\check{P}_1(1,t)^2+4D^2\dot{\hat{D}}(t)^2\check{P}_2(1,t)^2,\end{aligned} \quad (84)$$

and using (84) leads to

$$\begin{aligned}\dot{V}_1(t) \le& -A\left(2D-\frac{1}{2\iota_1}\right)\|w_x\|^2-\|z\|^2-\|z_x\|^2\\ &-\left(1-\frac{1}{2\iota_2}\right)z(0,t)^2-\left(1-\frac{A\iota_1}{10}-\frac{\iota_2}{2}\right)z_x(0,t)^2\\ &+4\tilde{D}(t)^2\check{P}_1(1,t)^2+4D^2\dot{\hat{D}}(t)^2\check{P}_2(1,t)^2\\ &+\tilde{D}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_1(0,t)\mathrm{d}x-z(0,t)\check{P}_1(0,t)\right.\\ &-2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)\mathrm{d}x\right)-\tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\ &+D\dot{\hat{D}}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_2(0,t)\mathrm{d}x-z(0,t)\check{P}_2(0,t)\right.\\ &-2\int_0^1(1+x)z(x,t)\check{P}_2(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{2x}(x,t)\mathrm{d}x\right),\end{aligned} \quad (85)$$

where, $2D-\frac{1}{2\iota_1}>0,\quad 1-\frac{1}{2\iota_2}>0,\quad 1-\frac{A\iota_1}{10}-\frac{\iota_2}{2}\ge 0$, leading to $\frac{1}{4D}<\iota_1,\quad \frac{1}{2}<\iota_2<2$ and $0<A\le\frac{10-5\iota_2}{\iota_1}$.

Hence, using Poincare's inequality, we arrive at

$$\begin{aligned}\dot{V}_1(t) \le& -\frac{1}{4}A\left(2D-\frac{1}{2\iota_1}\right)\|\check{w}\|^2-\|z\|^2-\|z_x\|^2\\ &-\left(1-\frac{1}{2\iota_2}\right)z(0,t)^2+4D^2\dot{\hat{D}}(t)^2\check{P}_2(1,t)^2\\ &+4\tilde{D}(t)^2\check{P}_1(1,t)+\tilde{D}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_1(0,t)\mathrm{d}x\right.\\ &-z(0,t)\check{P}_1(0,t)-2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)\mathrm{d}x\right)-\tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\ &+D\dot{\hat{D}}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_2(0,t)\mathrm{d}x-z(0,t)\check{P}_2(0,t)\right.\\ &-2\int_0^1(1+x)z(x,t)\check{P}_2(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{2x}(x,t)\mathrm{d}x\right).\end{aligned} \quad (86)$$

Choosing $\eta=\min\left\{\frac{1}{4}A\left(2\underline{D}-\frac{1}{2\iota_1}\right),\ 1-\frac{1}{2\iota_2}\right\}$, and

$$V_0(t)=\|\check{w}\|^2+\|z\|^2+\|z_x\|^2+z(0,t)^2, \quad (87)$$

the following inequality holds

$$\begin{aligned}\dot{V}_1(t) \le& -\eta V_0(t)+4\tilde{D}(t)^2\check{P}_1(1,t)^2+4\bar{D}^2\dot{\hat{D}}(t)^2\check{P}_2(1,t)^2\\ &+\tilde{D}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_1(0,t)\mathrm{d}x-z(0,t)\check{P}_1(0,t)\right.\\ &-2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)\mathrm{d}x\right)-\tilde{D}(t)\frac{\dot{\hat{D}}(t)}{\theta}\\ &+D\dot{\hat{D}}(t)\left(2A\int_0^1 \check{w}(x,t)x\check{P}_2(0,t)\mathrm{d}x-z(0,t)\check{P}_2(0,t)\right.\\ &-2\int_0^1(1+x)z(x,t)\check{P}_2(x,t)\mathrm{d}x\\ &\left.-2\int_0^1(1+x)z_x(x,t)\check{P}_{2x}(x,t)\mathrm{d}x\right).\end{aligned} \quad (88)$$

With the help of Agmon's, Cauchy–Schwarz and Young's inequalities, and combining (59) and (60), one can perform quite long but simple calculations to derive the following estimates:

$$\check{P}_1(1,t)^2\le LV_0(t),\quad \check{P}_2(1,t)^2\le LV_0(t), \quad (89)$$
$$2A\int_0^1 \check{w}(x,t)x\check{P}_1(0,t)\mathrm{d}x+z(0,t)\check{P}_1(0,t)\le LV_0(t), \quad (90)$$
$$2A\int_0^1 \check{w}(x,t)x\check{P}_2(0,t)\mathrm{d}x+z(0,t)\check{P}_2(0,t)\le LV_0(t), \quad (91)$$
$$2\int_0^1(1+x)z(x,t)\check{P}_1(x,t)\mathrm{d}x\le LV_0(t), \quad (92)$$
$$2\int_0^1(1+x)z_x(x,t)\check{P}_{1x}(x,t)\mathrm{d}x\le LV_0(t), \quad (93)$$
$$2\int_0^1(1+x)z(x,t)\check{P}_2(x,t)\mathrm{d}x\le LV_0(t), \quad (94)$$
$$2\int_0^1(1+x)z_x(x,t)\check{P}_{2x}(x,t)\mathrm{d}x\le LV_0(t), \quad (95)$$

where the parameter L is defined below

$$\begin{aligned}L=\max_{\underline{D}\le\hat{D}(t)\le\bar{D}}&\left\{3M_1(1,\hat{D}(t))^2+\int_0^1 M_2(1,y,\hat{D}(t))^2\mathrm{d}y,\right.\\ &3\int_0^1 M_2(1,y,\hat{D}(t))^2\mathrm{d}y,\ 3\int_0^1 M_3(1,y,\hat{D}(t))^2\mathrm{d}y,\\ &3\int_0^1 M_4(1,y,\hat{D}(t))^2\mathrm{d}y,\ 2(A+1)\left(\int_0^1 M_4(0,y,\hat{D}(t))^2\mathrm{d}y\right)^{\frac{1}{2}},\\ &(2A+1)\left[|M_1(0,\hat{D}(t))|+2\left(\int_0^1 M_2(0,y,\hat{D}(t))^2\mathrm{d}y\right)^{\frac{1}{2}}\right],\\ &2\left[\left(\int_0^1 M_1(x,\hat{D}(t))^2\mathrm{d}x\right)^{\frac{1}{2}}\right.\\ &\left.+2\left(\int_0^1\int_0^1 M_2(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\right],\\ &4\left[\left(\int_0^1\int_0^1 M_3(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\right.\\ &\left.+\left(\int_0^1\int_0^1 M_4(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\right],\end{aligned}$$

$$2\left[\left(\int_0^1 M_{1x}(x,\hat{D}(t))^2\mathrm{d}x\right)^{\frac{1}{2}}\right.$$
$$\left.+2\left(\int_0^1\int_0^1 M_{2x}(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\right],$$
$$2\left[\left(\int_0^1 M_3(x,x,\hat{D}(t))^2\mathrm{d}x\right)^{\frac{1}{2}}\right.$$
$$+\left(\int_0^1\int_0^1 M_{3x}(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\left.\left.+2\left(\int_0^1\int_0^1 M_{4x}(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)^{\frac{1}{2}}\right]\right\}. \tag{96}$$

Based on the property of the kernels $k(x,y)$ and $l(x,y)$, we have proved the boundedness of all the above terms in Appendix C.

Thus, using (61), (62), (89)–(95) together with the standard properties of the projection operator, we have

$$\dot{V}_1(t)\leq -\eta V_0(t)+4\bar{D}^2L^3\theta^2V_0(t)^3+4L\tilde{D}(t)^2V_0(t)+3\bar{D}L^2\theta V_0(t)^2$$
$$+4L|\tilde{D}(t)|V_0(t). \tag{97}$$

Moreover, from (77), knowing that

$$\tilde{D}(t)^2\leq 2\theta V_1(t)-\underline{D}\theta\min\{2A,1\}V_0(t), \tag{98}$$

and using Cauchy–Schwarz and Young's inequalities, we arrive at

$$\left|\tilde{D}(t)\right|\leq\frac{\varepsilon}{2}+\frac{\tilde{D}(t)^2}{2\varepsilon}\leq\frac{\varepsilon}{2}+\frac{\theta}{\varepsilon}V_1(t)-\frac{\underline{D}\theta}{2\varepsilon}\min\{2A,1\}V_0(t), \tag{99}$$

which finally leads to

$$\dot{V}_1(t)\leq-\left(\eta-8L\theta V_1(t)-4L\left(\frac{\varepsilon}{2}+\frac{\theta}{\varepsilon}V_1(t)\right)\right)V_0(t)$$
$$-\left(\frac{2\underline{D}L\theta\min\{2A,1\}}{\varepsilon}-4\bar{D}^2L^3\theta^2V_0(t)-3\bar{D}L^2\theta\right)V_0(t)^2. \tag{100}$$

Again, from (77), we have

$$\frac{\underline{D}}{2}\min\{2A,1\}V_0(t)\leq V_1(t). \tag{101}$$

Substituting (101) into (100), we derive the following estimate

$$\dot{V}_1(t)\leq-\left(\eta-8\theta LV_1(t)-4L\left(\frac{\varepsilon}{2}+\frac{\theta}{\varepsilon}V_1(t)\right)\right)V_0(t)$$
$$-\left(\frac{2\underline{D}L\theta\min\{2A,1\}}{\varepsilon}-\frac{8\bar{D}^2L^3\theta^2}{\underline{D}\min\{2A,1\}}V_1(t)\right.$$
$$\left.-3\bar{D}L^2\theta\right)V_0(t)^2. \tag{102}$$

Selecting ε as

$$\varepsilon\leq\min\left\{\frac{\eta}{2L},\frac{2\underline{D}\min\{2A,1\}}{3\bar{D}L}\right\}, \tag{103}$$

to ensure $\rho_1>0$ and restricting the initial conditions so that

$$V_1(0)\leq\rho_1, \tag{104}$$

where

$$\rho_1\triangleq\min\left\{\frac{\underline{D}\min\{2A,1\}}{8\bar{D}^2L^2\theta}\left(\frac{2\underline{D}\min\{2A,1\}}{\varepsilon}-3\bar{D}L\right),\right.$$
$$\left.\frac{\varepsilon(\eta-2L\varepsilon)}{4L\theta(2\varepsilon+1)}\right\}, \tag{105}$$

we obtain

$$\dot{V}_1\leq-\delta_1(t)V_0(t)-\delta_2(t)V_0(t)^2, \tag{106}$$

where

$$\delta_1(t)=\eta-8L\theta V_1(t)-4L\left(\frac{\varepsilon}{2}+\frac{\theta}{\varepsilon}V_1(t)\right), \tag{107}$$

$$\delta_2(t)=\frac{2\underline{D}L\theta\min\{2A,1\}}{\varepsilon}-\frac{8\bar{D}^2L^3\theta^2}{\underline{D}\min\{2A,1\}}V_1(t)-3\bar{D}L^2\theta, \tag{108}$$

are nonnegative functions if the initial conditions satisfy (104). Hence, $V_1(t)\leq V_1(0),\ \forall t\geq 0$.

Using (68), we get

$$\Psi(t)\leq\max\{r_1,r_2,r_3,r_4,1\}\ \left(\|\check{w}\|^2+\|z\|^2+\|z_x\|^2\right.$$
$$\left.+z(0,t)^2+\tilde{D}(t)^2\right),$$
$$\leq\frac{\max\{r_1,r_2,r_3,r_4,1\}}{\min\{A\underline{D},\frac{\underline{D}}{2},\frac{1}{2\theta}\}}V_1(t),$$
$$\triangleq\rho_2V_1(t)\leq\rho_2V_1(0), \tag{109}$$

where $\Psi(t)$ is defined in (64) and

$$\rho_2=\frac{\max\{r_1,r_2,r_3,r_4,1\}}{\min\{A\underline{D},\frac{\underline{D}}{2},\frac{1}{2\theta}\}}. \tag{110}$$

From (104) and (109), we obtain $\rho=\rho_1\rho_2$.

Using (69) and (77), it follows that

$$V_1(t)\leq A\bar{D}\|\check{w}\|^2+2\bar{D}\|z\|^2+2\bar{D}\|z_x\|^2+\frac{\bar{D}}{2}z(0,t)^2+\frac{\tilde{D}(t)^2}{2\theta},$$
$$\leq\max\{A\bar{D},2\bar{D}\}(\|\check{w}\|^2+\|z\|^2+\|z_x\|^2+z(0,t)^2)+\frac{\tilde{D}(t)^2}{2\theta},$$
$$\leq\max\{A\bar{D},2\bar{D}\}\max\{s_1,s_2,s_3,s_4\}\left(\|u\|^2+\|v\|^2\right.$$
$$\left.+\|v_x\|^2+v(0,t)^2\right)+\frac{\tilde{D}(t)^2}{2\theta}$$
$$\leq\max\left\{\max\{A\bar{D},2\bar{D}\}\max\{s_1,s_2,s_3,s_4\},\frac{1}{2\theta}\right\}\Psi(t), \tag{111}$$

and hence,

$$V_1(0)\leq\max\left\{\max\{A\bar{D},2\bar{D}\}\max\{s_1,s_2,s_3,s_4\},\frac{1}{2\theta}\right\}\Psi(0). \tag{112}$$

Combining (109) and (112), we get (65) with

$$R=\rho_2\max\left\{\max\{A\bar{D},2\bar{D}\}\max\{s_1,s_2,s_3,s_4\},\frac{1}{2\theta}\right\}, \tag{113}$$

which completes the local stability proof. ∎

Next, we prove the regulation of the distributed plant and actuator states $u(x,t)$ and $v(x,t)$ to complete the proof of Theorem 1.

4.2. Pointwise boundedness and regulation of the distributed state

From (77) and (102), we get the boundedness of $\|\check{w}\|$, $\|z\|$, $\|z_x\|$, $z(0,t)$ and $\tilde{D}(t)$. Moreover, from (68), we also get the boundedness of $\|u\|$, $\|v\|$, $\|v_x\|$ and $v(0,t)$. We will prove (66) and (67) in Theorem 1 by applying Lemma D.2 (Smyshlyaev & Krstic, 2010) to ensure the following facts:

(1) $\|\check{w}\|$, $\|z\|$, $\|z_x\|$, $z(0,t)$ and $\tilde{D}(t)$ are integrable in time,
(2) $\frac{\mathrm{d}}{\mathrm{d}t}(\|\check{w}\|^2)$, $\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\|z_x\|^2)$ are bounded,
(3) $\|w_x\|^2$ is bounded.

(1) Time integrability

Knowing that

$$\int_0^t\|\check{w}(\tau)\|^2\mathrm{d}\tau\leq\frac{1}{\inf_{0\leq\tau\leq t}\delta_1(\tau)}\int_0^t\delta_1(\tau)V_0(\tau)\mathrm{d}\tau, \tag{114}$$

and using (107), the following equality is stated

$$\inf_{0\le\tau\le t}\delta_1(t)=\eta-8L\theta\rho_1-4L\left(\frac{\varepsilon}{2}+\frac{\theta}{\varepsilon}\rho_1\right). \tag{115}$$

Since $\dot V_1(t)\le-\delta_1(t)V_0(t)-\delta_2(t)V_0(t)^2$ and $\delta_2(t)$ is a nonnegative function, we have

$$\dot V_1\le-\delta_1(t)V_0(t). \tag{116}$$

Integrating (116) over $[0,\ t]$ leads to

$$\int_0^t\delta_1(\tau)V_0(\tau)\mathrm{d}\tau\le V_1(0)\le\rho_1. \tag{117}$$

Substituting (115) and (117) into (114), we get $\|\check w\|$ is square integrable in time. Similarly, one can establish that $\|z\|$, $\|z_x\|$, $z(0,t)$ and $\tilde D(t)$ are square-integrable in time. ■

(2) Boundedness of some functions' time derivatives

Define Lyapunov function

$$\begin{aligned}V_2(t)=&\frac{1}{2}\int_0^1\check w(x,t)^2\mathrm{d}x+b_1\frac{D}{2}\int_0^1(1+x)z(x,t)^2\mathrm{d}x\\&+b_1\frac{D}{2}\int_0^1(1+x)z_x(x,t)^2\mathrm{d}x,\end{aligned} \tag{118}$$

where b_1 is a positive constant. Taking the derivative of (118) with respect to time, we obtain

$$\begin{aligned}\dot V_2(t)=&\int_0^1\check w(x,t)\check w_t(x,t)\mathrm{d}x+b_1D\int_0^1(1+x)z(x,t)z_t(x,t)\mathrm{d}x\\&+b_1D\int_0^1(1+x)z_x(x,t)z_{xt}(x,t)\mathrm{d}x,\\=&\int_0^1\check w(x,t)\check w_{xx}(x,t)\mathrm{d}x+b_1E_2(t)+E_3(t)\\&+b_1\int_0^1(1+x)z(x,t)z_x(x,t)\mathrm{d}x\\&-b_1\tilde D(t)\int_0^1(1+x)z(x,t)\check P_1(x,t)\mathrm{d}x\\&-b_1\tilde D(t)\int_0^1(1+x)z_x(x,t)\check P_{1x}(x,t)\mathrm{d}x\\&-b_1D\dot{\hat D}(t)\int_0^1(1+x)z(x,t)\check P_2(x,t)\mathrm{d}x\\&-b_1D\dot{\hat D}(t)\int_0^1(1+x)z_x(x,t)\check P_{2x}(x,t)\mathrm{d}x,\end{aligned} \tag{119}$$

where

$$E_2(t)=\int_0^1(1+x)z_x(x,t)z_{xx}(x,t)\mathrm{d}x, \tag{120}$$

$$E_3(t)=-\int_0^1\check w(x,t)xz_t(0,t)\mathrm{d}x. \tag{121}$$

Using integration by parts we derive

$$\begin{aligned}E_2(t)=&z_x(1,t)^2-\frac{1}{2}z_x(0,t)^2-\frac{1}{2}\|z_x\|^2,\\\le&-\frac{1}{2}z_x(0,t)^2-\frac{1}{2}\|z_x\|^2+2|\tilde D(t)|^2\check P_1(1,t)^2\\&+2\bar D^2|\dot{\hat D}(t)|^2\check P_2(1,t)^2.\end{aligned} \tag{122}$$

By using Cauchy–Schwarz and Young's inequalities, the following holds

$$\begin{aligned}E_3(t)\le&\frac{1}{2\iota_3}\|\check w\|^2+\frac{\iota_3}{6}z_t(0,t)^2,\\\le&\frac{1}{2\iota_3}\|\check w\|^2+\frac{\iota_3}{2\underline{D}^2}z_x(0,t)^2+\frac{\iota_3}{2\underline{D}^2}|\tilde D(t)|^2\check P_1(0,t)^2\\&+\frac{\iota_3}{2}|\dot{\hat D}(t)|^2\check P_2(0,t)^2.\end{aligned} \tag{123}$$

Hence, using Cauchy–Schwarz inequality and combining (122), (123), we deduce the following inequality

$$\begin{aligned}\dot V_2(t)\le&-\|\check w_x\|^2+\frac{1}{2\iota_3}\|\check w\|^2-\frac{b_1}{2}\|z\|^2-\frac{b_1}{2}\|z_x\|^2-\frac{b_1}{2}z(0,t)^2\\&-\left(\frac{b_1}{2}-\frac{\iota_3}{2\underline{D}^2}\right)z_x(0,t)^2+2b_1|\tilde D(t)|\|z\|\|\check P_1\|\\&+2b_1\bar D|\dot{\hat D}(t)|\|z\|\|\check P_2\|+2b_1|\tilde D(t)|\|z_x\|\|\check P_{1x}\|\\&+2b_1\bar D|\dot{\hat D}(t)|\|z_x\|\|\check P_{2x}\|+\frac{\iota_3}{2\underline{D}^2}|\tilde D(t)|^2\check P_1(0,t)^2\\&+\frac{\iota_3}{2}|\dot{\hat D}(t)|^2\check P_2(0,t)^2+2b_1|\tilde D(t)|^2\check P_1(1,t)^2\\&+2b_1\bar D^2|\dot{\hat D}(t)|^2\check P_2(1,t)^2.\end{aligned} \tag{124}$$

Next, selecting $\iota_3=4$ and $b_1>\frac{4}{\underline{D}^2}$, we have

$$\begin{aligned}\dot V_2(t)\le&-\frac{1}{8}\|\check w\|^2-\frac{b_1}{2}\|z\|^2-\frac{b_1}{2}\|z_x\|^2+b_1\bar D^2|\dot{\hat D}(t)|^2\|z\|^2\\&+b_1|\tilde D(t)|^2\|z\|^2+b_1\|\check P_1\|^2+b_1\bar D^2|\dot{\hat D}(t)|^2\|z_x\|^2\\&+b_1|\tilde D(t)|^2\|z_x\|^2+b_1\|\check P_2\|^2+b_1\|\check P_{1x}\|^2+b_1\|\check P_{2x}\|^2\\&+\frac{2}{\underline{D}^2}|\tilde D(t)|^2\check P_1(0,t)^2+2|\dot{\hat D}(t)|^2\check P_2(0,t)^2\\&+2b_1|\tilde D(t)|^2\check P_1(1,t)^2+2b_1\bar D^2|\dot{\hat D}(t)|^2\check P_2(1,t)^2,\\\le&-c_1V_2+f_1(t)V_2+f_2(t),\end{aligned} \tag{125}$$

where we employ Poincare's and Young's inequalities. Here, $c_1=\min\left\{\frac{1}{4},\frac{1}{2\bar D}\right\}$ and $f_i(t),\ i=\{1,2\}$ are defined as

$$f_1(t)=\frac{2}{\underline{D}}(\bar D^2|\dot{\hat D}(t)|^2+|\tilde D(t)|^2), \tag{126}$$

$$\begin{aligned}f_2(t)=&\frac{2}{\underline{D}^2}|\tilde D(t)|^2\check P_1(0,t)^2+2b_1|\tilde D(t)|^2\check P_1(1,t)^2+b_1\|\check P_1\|^2\\&+2|\dot{\hat D}(t)|^2\check P_2(0,t)^2+2b_1\bar D^2|\dot{\hat D}(t)|^2\check P_2(1,t)^2\\&+b_1\|\check P_2\|^2+b_1\|\check P_{1x}\|^2+b_1\|\check P_{2x}\|^2.\end{aligned} \tag{127}$$

Combining (59), (60) with (89)–(95), we get that $|\dot{\hat D}(t)|$, $\check P_1(0,t)^2$, $\check P_1(1,t)^2$, $\|\check P_1\|^2$, $\check P_2(0,t)^2$, $\check P_2(1,t)^2$, $\|\check P_2\|^2$, $\|\check P_{1x}\|^2$, $\|\check P_{2x}\|^2$ are bounded and integrable. Thereby, $f_1(t)$ and $f_2(t)$ are bounded and integrable functions of time. Thus, from (125), we deduce that $\dot V_2\le\infty$, which proves the boundedness of $\frac{\mathrm{d}}{\mathrm{d}t}(\|\check w\|^2)$, $\frac{\mathrm{d}}{\mathrm{d}t}(\|z\|^2)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\|z_x\|^2)$. Moreover, by Lemma D.2 (Smyshlyaev & Krstic, 2010), it holds that $\|\check w\|,\|z\|,\|z_x\|\to0$ as $t\to\infty$. Knowing that $z(0,t)\le2\|z\|\|z_x\|$, so $z(0,t)\to0$ as $t\to\infty$. ■

(3) Boundedness of $\|w_x\|$

Define the following Lyapunov function

$$V_3=\frac{1}{2}\int_0^1\check w_x(x,t)^2\mathrm{d}x+b_2\frac{D}{2}\int_0^1(1+x)z_x(x,t)^2\mathrm{d}x, \tag{128}$$

where b_2 is a positive constant. Using the integration by parts, the derivative of (128) with respect to time is written as

$$\begin{aligned}\dot V_3(t)=&\int_0^1\check w_x(x,t)\check w_{xt}(x,t)\mathrm{d}x+b_2D\int_0^1(1+x)z_x(x,t)z_{xt}(x,t)\mathrm{d}x,\\=&E_4(t)+b_2E_2(t)-b_2\tilde D(t)\int_0^1(1+x)z_x(x,t)\check P_{1x}(x,t)\mathrm{d}x\\&-b_2D\dot{\hat D}(t)\int_0^1(1+x)z_x(x,t)\check P_{2x}(x,t)\mathrm{d}x,\end{aligned} \tag{129}$$

where $E_2(t)$ is defined in (120), and

$$E_4(t)=-\int_0^1\check w_{xx}(x,t)\check w_t(x,t)\mathrm{d}x. \tag{130}$$

Next, we derive the following estimate:

$$\begin{aligned}E_4(t) =& -\int_0^1 \check{w}_{xx}^2(x,t)\mathrm{d}x + \int_0^1 \check{w}_{xx} x z_t(0,t)\mathrm{d}x,\\ \le& -\|\check{w}_{xx}\|^2 + \frac{1}{2\iota_4}\|\check{w}_{xx}\|^2 + \frac{\iota_4}{6} z_t(0,t)^2,\\ \le& -\left(1-\frac{1}{2\iota_4}\right)\|\check{w}_{xx}\|^2 + \frac{\iota_4}{2\underline{D}^2}|\tilde{D}(t)|^2\check{P}_2(0,t)^2\\ &+\frac{\iota_4}{2\underline{D}^2} z_x(0,t)^2 + \frac{\iota_4}{2}|\dot{\hat{D}}(t)|^2\check{P}_1(0,t)^2, \end{aligned} \tag{131}$$

where we have used Cauchy–Schwarz and Young's inequalities. Substituting (122) and (131) into (129), and again, using Cauchy–Schwarz inequality, we arrive at the following inequality

$$\begin{aligned}\dot{V}_3(t) \le& -\left(1-\frac{1}{2\iota_4}\right)\|\check{w}_{xx}\|^2 - \frac{b_2}{2}\|z_x\|^2 - \left(\frac{b_2}{2}-\frac{\iota_4}{2\underline{D}^2}\right) z_x(0,t)^2\\ &+\frac{\iota_4}{2\underline{D}^2}|\tilde{D}(t)|^2\check{P}_1(0,t)^2 + \frac{\iota_4}{2}|\dot{\hat{D}}(t)|^2\check{P}_2(0,t)^2\\ &+2b_2|\tilde{D}(t)|^2\check{P}_1(1,t)^2 + 2b_2\bar{D}^2|\dot{\hat{D}}(t)|^2\check{P}_2(1,t)^2\\ &+2b_2|\tilde{D}(t)|\|z_x\|\|\check{P}_{1x}\| + 2b_2\bar{D}|\dot{\hat{D}}(t)|\|z_x\|\|\check{P}_{2x}\|. \end{aligned} \tag{132}$$

Next, choosing $\iota_4 = 1$ and $b_2 > \frac{1}{\underline{D}^2}$, we get

$$\begin{aligned}\dot{V}_3(t) \le& -\frac{1}{8}\|\check{w}_x\|^2 - \frac{b_2}{2}\|z_x\|^2 + b_2\bar{D}^2|\dot{\hat{D}}(t)|^2\|z_x\|^2\\ &+b_2\|\check{P}_{1x}\|^2 + b_2\bar{D}^2\|z_x\|^2 + b_2\|\check{P}_{2x}\|^2\\ &+\frac{1}{2\underline{D}^2}|\tilde{D}(t)|^2\check{P}_1(0,t)^2 + \frac{1}{2}|\dot{\hat{D}}(t)|^2\check{P}_2(0,t)^2\\ &+2b_2|\tilde{D}(t)|^2\check{P}_1(1,t)^2 + 2b_2\bar{D}^2|\dot{\hat{D}}(t)|^2\check{P}_2(1,t)^2,\\ \le& -c_1V_3 + f_1(t)V_3 + f_3(t), \end{aligned} \tag{133}$$

where we use Poincare's and Young's inequalities. We recall that $f_1(t)$ defined in (126) is bounded, and

$$\begin{aligned}f_3(t) =& b_2\|\check{P}_{1x}\|^2 + b_2\|\check{P}_{2x}\|^2 + \frac{1}{2\underline{D}^2}|\tilde{D}(t)|^2\check{P}_1(0,t)^2\\ &+\frac{1}{2}|\dot{\hat{D}}(t)|^2\check{P}_2(0,t)^2 + 2b_2|\tilde{D}(t)|^2\check{P}_1(1,t)^2\\ &+2b_2\bar{D}^2|\dot{\hat{D}}(t)|^2\check{P}_2(1,t)^2, \end{aligned} \tag{134}$$

is bounded and integrable in time. Using Lemma D.3 (Smyshlyaev & Krstic, 2010), we get that $\|\check{w}_x\|$ is bounded which implies the boundedness of $\|w_x\|$ based on the following relation

$$\|w_x\|^2 \le 2\|\check{w}_x\|^2 + 2z(0,t)^2, \tag{135}$$

that is possible to infer from (54). ■

Proof of (66) and (67) in Theorem 1. Finally, since $\|\check{w}\|$, $\|z\|$, $\|z_x\|$, $z(0,t) \to 0$ as $t \to \infty$, from (68), we have $\|u\|^2$, $\|v\|^2$, $\|v_x\|^2$, $v(0,t)^2 \to 0$ as $t \to \infty$. Then, knowing that $\|w_x\|$ is bounded, from (27) and (37), we deduce that

$$\|u_x\|^2 \le 2(3\lambda^2+7)\|w_x\|^2 \tag{136}$$

is bounded, as well. By Agmon's inequality, one can state that $u(x,t)^2 \le 2\|u\|\|u_x\|$, which proves (66). Likewise, one can derive the proof of (67), which completes the proof of Theorem 1. ■

5. Simulation results

To illustrate the feasibility of the proposed adaptive boundary controller design, we simulate the closed-loop system consisting of (5)–(8), the control law (40), and the update law (61)–(63). The value of the delay is set to $D = 1$, assuming the known upper and lower bounds as $\bar{D} = 2$ and $\underline{D} = 0.1$, respectively. The adaptation gain is set to $\theta = 0.71$, and the plant reaction coefficient is chosen as $\lambda = 10$. For two distinct initial values of the delay, namely, $\hat{D}(0) = 0.1$ and $\hat{D}(0) = 2$, the simulations are performed considering $u(x,0) = u_0(x) = \cos(2\pi x)$ as the distributed plant's state at the initial time $t = 0$. The initial condition of the distributed actuator state is set to $v_0(x) = \cos(2\pi x)$. The numerical values of the parameters given in Theorem (68) are $\rho = 31.6699$ and $R = 3.0302 \times 10^{35}$. Moreover, from the initial data, $\Psi(0) = 22.5492$ and $\Psi(0) = 22.7392$ for $\hat{D}(0) = 0.1$ and $\hat{D}(0) = 2$, respectively, which is consistent with the inequality $\Psi(0) < \rho$.

Fig. 1 shows the convergence of the plant's state. In the absence of adaptation, but with a "mismatch input delay" set to $\hat{D}(t) = 2$ (the true delay being $D = 1$), the control law (40) results into a slower convergence. Fig. 2 (a) shows the dynamics of the L^2-norm of the distributed state with and without adaption. Clearly, an adaptive controller allows faster convergence than a nonadaptive scheme irrespective of initial conditions. Fig. 2(b) presents the dynamics of the control effort, while Fig. 2(c) illustrates the evolution of the update law. Last but not least, Fig. 2(d) reflects a good estimate of the delay with $\hat{D}(t)$, which converges to the true value $D = 1$.

6. Conclusion

In this paper, we design an adaptive controller and a delay parameter's update law to stabilize a scalar reaction–diffusion system with an arbitrarily large and unknown boundary input delay. Based on a Lyapunov argument, the adaptively controlled plant is locally L^2-stable. Numerical simulations are presented to support the theoretical statements. Further research includes the designing of an observer for the case of an unmeasurable actuator state combined with the design of a controller that ensures a global stability result.

Appendix A. Proof the boundedness of kernels

The proof of the boundedness of all kernels is established using the following Lemmas.

Lemma 1. *The kernel $\eta(x,y,\hat{D}(t))$ satisfies the following diffusion PDE:*

$$\eta_x(x,y,\hat{D}(t)) = \hat{D}(t)\eta_{yy}(x,y,\hat{D}(t)), \tag{A.1}$$

$$\eta(x,1,\hat{D}(t)) = \eta(x,0,\hat{D}(t)) = 0, \tag{A.2}$$

and the following hold:

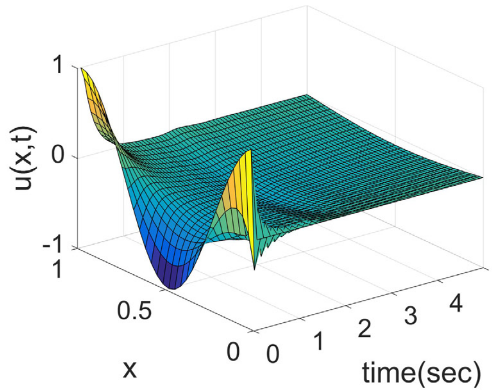
$$\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x \le \frac{2\left(1-e^{-\frac{\pi^2}{2}\bar{D}}\right)}{\underline{D}\pi^2}\int_0^1 l(1,y)^2\mathrm{d}y, \tag{A.3}$$

$$\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x \le \frac{2\bar{D}\left(1-e^{-\frac{\pi^2}{2}\bar{D}}\right)}{\pi^2}\int_0^1 l_{yy}(1,y)^2\mathrm{d}y, \tag{A.4}$$

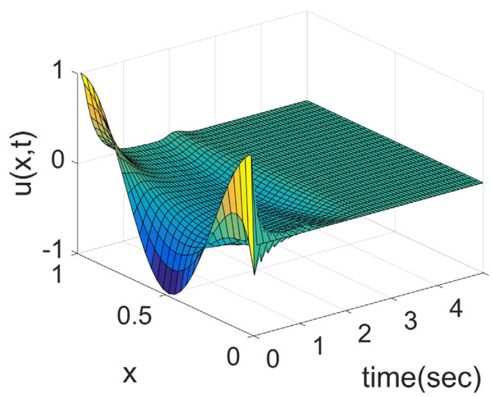
where

$$l(1,y) = -\lambda y\frac{J_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}, \tag{A.5}$$

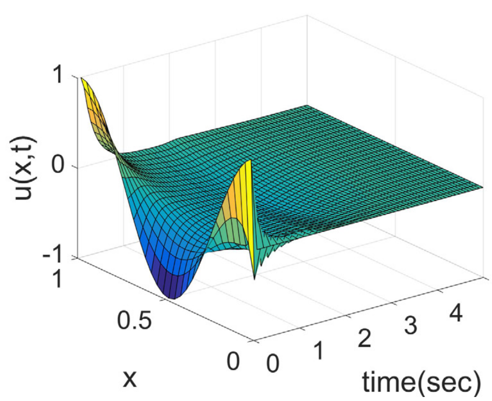
$$l_{yy}(1,y) = -3\lambda^2 y\frac{J_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)} - \lambda^3y^3\frac{J_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3}, \tag{A.6}$$



(a) The distributed state $u(x,t)$ with nonadaptive control and $D=2$.

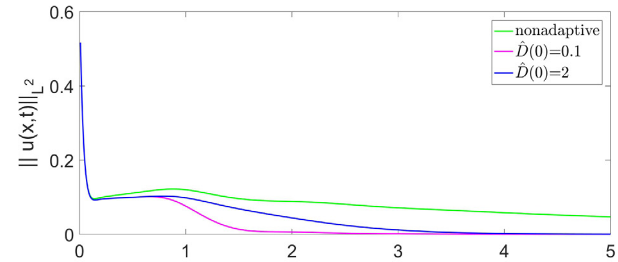


(b) The distributed state $u(x,t)$ with $\hat{D}(0)=0.1$.

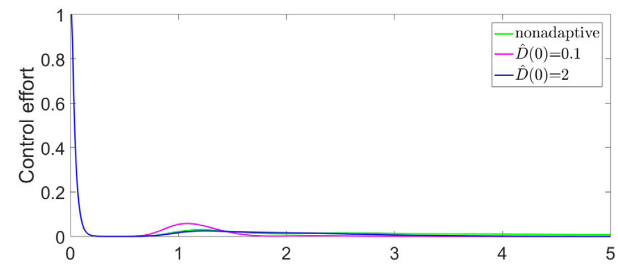


(c) The distributed state $u(x,t)$ with $\hat{D}(0)=2$.

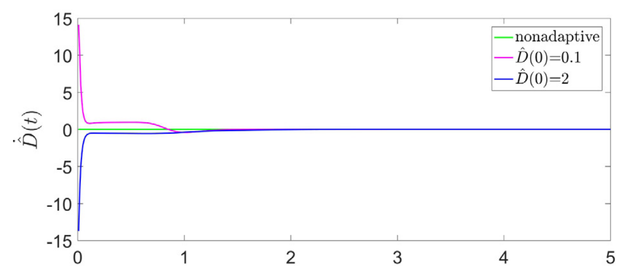
Fig. 1. The distributed state dynamics under closed-loop control.



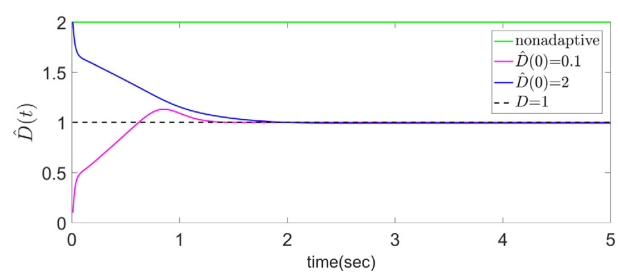
(a) L^2-norm of the distributed state $u(x,t)$.



(b) The time-evolution of the control signal.



(c) The dynamics of the update law $\hat{D}(t)$.



(d) The time-evolution of the estimate of the unknown parameter $\hat{D}(t)$.

Fig. 2. The closed-loop system dynamics with and without adaptation.

are continuous functions with

$$l(1,1) = -\frac{\lambda}{2}, \quad l_{yy}(1,1) = -\frac{\lambda^3}{48} - \frac{3\lambda^2}{8}. \tag{A.7}$$

Proof. To prove (A.3), we use the following fact

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y &= \int_0^1 \eta(x,y,\hat{D}(t))\eta_x(x,y,\hat{D}(t))\mathrm{d}y,\\ &= \hat{D}(t)\int_0^1 \eta(x,y,\hat{D}(t))\eta_{yy}(x,y,\hat{D}(t))\mathrm{d}y,\\ &= -\hat{D}(t)\int_0^1 \eta_y(x,y,\hat{D}(t))^2\mathrm{d}y,\\ &\le -\hat{D}(t)\frac{\pi^2}{4}\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y,\end{aligned} \tag{A.8}$$

where we use integration by parts and the Wirtinger inequality. Hence, by the comparison principle

$$\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y \le e^{-\frac{\pi^2}{2}\hat{D}(t)x}\int_0^1 \eta(0,y,\hat{D}(t))^2\mathrm{d}y. \tag{A.9}$$

Knowing that $\eta(0,y,\hat{D}(t)) = l(1,y)$, we get

$$\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y \le e^{-\frac{\pi^2}{2}\hat{D}(t)x}\int_0^1 l(1,y)^2\mathrm{d}y, \tag{A.10}$$

and integrating (A.10) with respect to x, we obtain

$$\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2 dydx \le \int_0^1 e^{-\frac{\pi^2}{2}\hat{D}(t)x}dx\int_0^1 l(1,y)^2dy, \le \frac{2\left(1-e^{-\frac{\pi^2}{2}\hat{D}(t)}\right)}{\hat{D}(t)\pi^2}\int_0^1 l(1,y)^2dy. \quad (A.11)$$

Finally, from the boundedness of $\hat{D}(t)$, inequality (A.3) is deduced. Using a similar calculation as for proving inequality (A.3), one can easily prove (A.4). ■

Lemma 2. *The kernel $\eta(x,y,\hat{D}(t))$ satisfies the diffusion PDE (A.1), (A.2) and the following hold:*

$$\int_0^1 \eta_y(x,1,\hat{D}(t))^2dx \le \frac{1}{2}\int_0^1 l_y(1,y)^2dy + \frac{(\bar{D}+1)}{2\underline{D}^2}\int_0^1 l(1,y)^2dy, \quad (A.12)$$

$$\int_0^1 \eta_{xy}(x,1,\hat{D}(t))^2dx \le \frac{\bar{D}^2}{2}\int_0^1 l_{yyy}(1,y)^2dy + \frac{(\bar{D}+1)}{2}\int_0^1 l_{yy}(1,y)^2dy, \quad (A.13)$$

with

$$l_y(1,y) = -\lambda\frac{J_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} - \lambda^2y^2\frac{J_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)}, \quad (A.14)$$

$$l_{yyy}(1,y) = -3\lambda^2\frac{J_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)} - 6\lambda^3y^2\frac{J_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3} - \lambda^4y^4\frac{J_4\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\lambda(1-y^2)\right)^2}, \quad (A.15)$$

being continuous functions and,

$$l_y(1,1) = -\frac{\lambda^2}{8}-\frac{\lambda}{2}, \quad (A.16)$$

$$l_{yyy}(1,1) = -\frac{\lambda^4}{384}-\frac{\lambda^3}{8}-\frac{3\lambda^2}{8}. \quad (A.17)$$

Proof. To prove (A.12), we multiply the PDE

$$\eta_x(x,y,\hat{D}(t)) = \hat{D}(t)\eta_{yy}(x,y,\hat{D}(t)) \quad (A.18)$$

by $2y\eta_y(x,y,\hat{D}(t))$ to arrive at the equality below

$$2y\eta_x(x,y,\hat{D}(t))\eta_y(x,y,\hat{D}(t)) = 2\hat{D}(t)y\eta_y(x,y,\hat{D}(t))\eta_{yy}(x,y,\hat{D}(t)). \quad (A.19)$$

Integrating (A.19) with respect to y, and using integration by parts we get

$$2\int_0^1 y\eta_x(x,y,\hat{D}(t))\eta_y(x,y,\hat{D}(t))dy = \hat{D}(t)\eta_y(x,1,\hat{D}(t))^2 - \hat{D}(t)\int_0^1 \eta_y(x,y,\hat{D}(t))^2dy. \quad (A.20)$$

By using Cauchy–Schwarz and Young's inequalities, the following holds

$$\eta_y(x,1,\hat{D}(t))^2 \le \frac{1}{\hat{D}(t)}\int_0^1 \eta_x(x,y,\hat{D}(t))^2dy + \frac{(\hat{D}(t)+1)}{\hat{D}(t)}\int_0^1 \eta_y(x,y,\hat{D}(t))^2dy. \quad (A.21)$$

Then, integrating (A.21) with respect to x we arrive at

$$\int_0^1 \eta_y(x,1,\hat{D}(t))^2dx \le \frac{1}{\hat{D}(t)}\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2dydx + \frac{(\hat{D}(t)+1)}{\hat{D}(t)}\int_0^1\int_0^1 \eta_y(x,y,\hat{D}(t))^2dydx. \quad (A.22)$$

On the right side of the above inequality, the double integral terms are bounded as follows

$$\int_0^1\int_0^1 \eta_y(x,y,\hat{D}(t))^2dydx \le \frac{1}{2\hat{D}(t)}\int_0^1 l(1,y)^2dy, \quad (A.23)$$

$$\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2dydx \le \frac{\hat{D}(t)}{2}\int_0^1 l_y(1,y)^2dy. \quad (A.24)$$

Here, we prove (A.24) as an example. From (A.1)–(A.2), we have,

$$\frac{d}{dx}\frac{1}{2}\int_0^1 \eta_x(x,y,\hat{D}(t))^2dy = -\hat{D}(t)\int_0^1 \eta_{yy}(x,y,\hat{D}(t))^2dy. \quad (A.25)$$

Now, integrating (A.25) with respect to x, we obtain

$$\int_0^1\int_0^1 \eta_{yy}(x,y,\hat{D}(t))^2dydx \le \frac{1}{2\hat{D}(t)}\int_0^1 \eta_y(0,y,\hat{D}(t))^2dy. \quad (A.26)$$

From (A.1) and the fact that $\eta_y(0,y,\hat{D}(t)) = l_y(1,y,\hat{D}(t))$, one can straightforwardly deduce (A.24). Substituting (A.23) and (A.24) into (A.22), one can derive (A.12). Likewise, one can easily get (A.13) and complete the proof of the lemma. ■

In an analogous way, we state the following two lemmas which can be proved employing the method presented above.

Lemma 3. *The kernel $\gamma(x,y,\hat{D}(t))$ satisfies the following reaction–diffusion PDE:*

$$\gamma_x(x,y,\hat{D}(t)) = \hat{D}(t)\gamma_{yy}(x,y,\hat{D}(t)) + \hat{D}(t)\lambda\gamma(x,y,\hat{D}(t)), \quad (A.27)$$

$$\gamma(x,1,\hat{D}(t)) = \gamma(x,0,\hat{D}(t)) = 0, \quad (A.28)$$

and the following hold:

$$\int_0^1\int_0^1 \gamma(x,y,\hat{D}(t))^2dydx \le \frac{2\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{\underline{D}(4\lambda-\pi^2)}\int_0^1 k(1,y)^2dy, \quad (A.29)$$

$$\int_0^1\int_0^1 \gamma_x(x,y,\hat{D}(t))^2dydx \le \frac{4\bar{D}\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{4\lambda-\pi^2}\int_0^1 (k_{yy}(1,y)^2+\lambda^2k(1,y)^2)dy, \quad (A.30)$$

$$\int_0^1\int_0^1 \gamma_{xx}(x,y,\hat{D}(t))^2dydx \le \frac{6\bar{D}^3\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{4\lambda-\pi^2}\int_0^1 \left(k_{yyyy}(1,y)^2+4\lambda^2k_{yy}(1,y)^2 + \lambda^4k(1,y)^2\right)dy, \quad (A.31)$$

where

$$k(1,y) = -\lambda y\frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}, \quad (A.32)$$

$$k_{yy}(1,y)=3\lambda^2y\frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)}-\lambda^3y^3\frac{I_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3},\tag{A.33}$$

$$\begin{aligned}k_{yyyy}(1,y)=&-15\lambda^3y\frac{I_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3}\\&+10\lambda^4y^3\frac{I_4\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\lambda(1-y^2)\right)^2}-\lambda^5y^5\frac{I_5\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^5},\end{aligned}\tag{A.34}$$

are continuous functions with

$$k(1,1)=-\frac{\lambda}{2},\quad k_{yy}(1,1)=-\frac{\lambda^3}{48}+\frac{3\lambda^2}{8},\tag{A.35}$$

$$k_{yyyy}(1,1)=-\frac{\lambda^5}{3840}+\frac{5\lambda^4}{192}-\frac{5\lambda^3}{16}.\tag{A.36}$$

Lemma 4. *The kernel $\gamma(x,y,\hat{D}(t))$ satisfies the reaction–diffusion PDE (A.27)–(A.28) and the following hold:*

$$\begin{aligned}&\int_0^1\gamma_y(x,1,\hat{D}(t))^2dx\\&\le\left(\frac{2\lambda(4\bar{D}^2\lambda+2\bar{D}+1)\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{\underline{D}^2(4\lambda-\pi^2)}+\lambda+\frac{\bar{D}+1}{2\underline{D}^2}\right)\\&\cdot\int_0^1k(1,y)^2dy+\int_0^1k_y(1,y)^2dy,\end{aligned}\tag{A.37}$$

$$\begin{aligned}&\int_0^1\gamma_{yx}(x,1,\hat{D}(t))^2dx\\&\le(4\bar{D}^2\lambda+2\bar{D}+1)\frac{8\lambda^3\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{4\lambda-\pi^2}\int_0^1k(1,y)^2dy\\&+(2\bar{D}^2\lambda+\bar{D}+1)\int_0^1(k_{yy}(1,y)^2+\lambda^2k(1,y)^2)dy\\&+\lambda(4\bar{D}^2\lambda+2\bar{D}+1)\int_0^1(k_y(1,y)^2+\lambda k(1,y)^2)dy\\&+2\bar{D}^2\int_0^1(k_{yyy}(1,y)^2+\lambda^2k_y(1,y)^2)dy,\end{aligned}\tag{A.38}$$

$$\begin{aligned}&\int_0^1\gamma_{yxx}(x,1,\hat{D}(t))^2dx\\&\le(4\bar{D}^2\lambda+2\bar{D}+1)\frac{32\bar{D}^2\lambda^5\left(e^{\bar{D}(2\lambda-\frac{\pi^2}{2})}-1\right)}{4\lambda-\pi^2}\int_0^1k(1,y)^2dy\\&+2\bar{D}^2\lambda(4\bar{D}^2\lambda+2\bar{D}+1)\int_0^1\Big(k_{yyy}(1,y)^2+\lambda k_{yy}(1,y)^2\\&+3\lambda^2k_y(1,y)^2+3\lambda^3k(1,y)^2\Big)dy+\frac{3\bar{D}^2}{2}(2\bar{D}^2\lambda+\bar{D}+1)\\&\cdot\int_0^1\left(k_{yyyy}(1,y)^2+4\lambda^2k_{yy}(1,y)^2+\lambda^4k(1,y)^2\right)dy\\&+3\bar{D}^4\int_0^1\left(k_{yyyyy}(1,y)^2+4\lambda^2k_{yyy}(1,y)^2+\lambda^4k_y(1,y)^2\right)dy,\end{aligned}\tag{A.39}$$

where

$$k_y(1,y)=-\lambda\frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}}+\lambda^2y^2\frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)},\tag{A.40}$$

$$\begin{aligned}k_{yyy}(1,y)=&\,3\lambda^2\frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{\lambda(1-y^2)}-6\lambda^3y^2\frac{I_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3}\\&+\lambda^4y^4\frac{I_4\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\lambda(1-y^2)\right)^2},\end{aligned}\tag{A.41}$$

$$\begin{aligned}k_{yyyyy}(1,y)=&-15\lambda^3\frac{I_3\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^3}\\&+45\lambda^4y^2\frac{I_4\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\lambda(1-y^2)\right)^2}-15\lambda^5y^4\frac{I_5\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\sqrt{\lambda(1-y^2)}\right)^5}\\&+\lambda^6y^6\frac{I_6\left(\sqrt{\lambda(1-y^2)}\right)}{\left(\lambda(1-y^2)\right)^3},\end{aligned}\tag{A.42}$$

are continuous functions with

$$k_y(1,1)=\frac{\lambda^2}{8}-\frac{\lambda}{2},\quad k_{yyy}(1,1)=\frac{\lambda^4}{384}-\frac{\lambda^3}{8}+\frac{3\lambda^2}{8},\tag{A.43}$$

$$k_{yyyyy}(1,1)=\frac{\lambda^6}{46080}-\frac{\lambda^5}{256}+\frac{15\lambda^4}{128}-\frac{5\lambda^3}{16}.\tag{A.44}$$

Appendix B. Proof of Proposition 1

To prove Proposition 1, we derive the following estimate of the L^2 norm of u:

$$\begin{aligned}\int_0^1u(x,t)^2dx=&\int_0^1\Big(\check{w}(x,t)+\int_0^xl(x,y)\check{w}(y,t)dy\\&+xz(0,t)+\int_0^xl(x,y)yz(0,t)dy\Big)^2dx,\\\le&\,4\int_0^1\check{w}(x,t)^2dx+4\int_0^1\left(\int_0^xl(x,y)\check{w}(y,t)dy\right)^2dx\\&+\frac{4}{3}z(0,t)^2+4\int_0^1\left(\int_0^xl(x,y)yz(0,t)dy\right)^2dx,\\\le&\,4\left(1+\int_0^1\int_0^xl(x,y)^2dydx\right)\|\check{w}\|^2\\&+\frac{1}{3}z(0,t)^2\left(4+\int_0^1\int_0^xl(x,y)^2dydx\right),\end{aligned}\tag{B.1}$$

where we use Cauchy–Schwarz and Young's inequalities. Then, the L^2 norm of v satisfies the following relations

$$\begin{aligned}\int_0^1v(x,t)^2dx=&\int_0^1\Big(z(x,t)+\hat{D}(t)\int_0^xp(x,y,\hat{D}(t))z(y,t)dy\\&+\int_0^1\eta(x,y,\hat{D}(t))(\check{w}(y,t)+yz(0,t))dy\Big)^2dx,\\\le&\,4\int_0^1z(x,t)^2dx+4\int_0^1\left(\int_0^1\eta(x,y,\hat{D}(t))\check{w}(y,t)dy\right)^2dx\\&+4\int_0^1\left(\int_0^1\eta(x,y,\hat{D}(t))yz(0,t)dy\right)^2dx\end{aligned}$$

$$
\begin{aligned}
&+4\bar{D}^2\int_0^1\left(\int_0^x p(x,y,\hat{D}(t))z(y,t)\mathrm{d}y\right)^2\mathrm{d}x,\\
&\le 4\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\|\check{w}\|^2\\
&+4\left(1+\bar{D}^2\int_0^1\int_0^x \eta_y(x-y,1,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)\|z\|^2\\
&+\frac{4}{3}z(0,t)^2\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x, \qquad \text{(B.2)}
\end{aligned}
$$

where we use Cauchy–Schwarz and Young's inequalities, and the fact that $p(x,y,\hat{D}(t)) = -\eta_y(x-y,1,\hat{D}(t))$. Since

$$
\begin{aligned}
&\int_0^1\int_0^x \eta_y(x-y,1,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^x \eta_y(s,1,\hat{D}(t))^2\mathrm{d}s\mathrm{d}x\\
&\le \int_0^1\int_0^1 \eta_y(s,1,\hat{D}(t))^2\mathrm{d}s\mathrm{d}x \le \int_0^1 \eta_y(x,1,\hat{D}(t))^2\mathrm{d}x, \qquad \text{(B.3)}
\end{aligned}
$$

it is clear that

$$
\begin{aligned}
\int_0^1 v(x,t)^2\mathrm{d}x \le{}& \frac{4}{3}z(0,t)^2\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\\
&+4\left(1+\bar{D}^2\int_0^1 \eta_y(x,1,\hat{D}(t))^2\mathrm{d}x\right)\|z\|^2\\
&+4\int_0^1\int_0^1 \eta(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\|\check{w}\|^2. \qquad \text{(B.4)}
\end{aligned}
$$

Next, considering the first derivative of $v(x,t)$ with respect to x, we derive the following estimates

$$
\begin{aligned}
&\int_0^1 v_x(x,t)^2\mathrm{d}x = \int_0^1\left(z_x(x,t)+\int_0^1 \eta_x(x,y,\hat{D}(t))\check{w}(y,t)\mathrm{d}y\right.\\
&+\int_0^1 \eta_x(x,y,\hat{D}(t))yz(0,t)\mathrm{d}y + \hat{D}(t)p(x,x,\hat{D}(t))z(x,t)\\
&\left.+\hat{D}(t)\int_0^x p_x(x,y,\hat{D}(t))z(y,t)\mathrm{d}y\right)^2\mathrm{d}x,\\
&\le 5\int_0^1 z_x(x,t)^2\mathrm{d}x + 5\int_0^1\left(\int_0^1 \eta_x(x,y,\hat{D}(t))\check{w}(y,t)\mathrm{d}y\right)^2\mathrm{d}x\\
&+5\int_0^1\left(\int_0^1 \eta_x(x,y,\hat{D}(t))yz(0,t)\mathrm{d}y\right)^2\mathrm{d}x\\
&+5\bar{D}^2\int_0^1 (p(x,x,\hat{D}(t))z(x,t))^2\mathrm{d}x\\
&+5\bar{D}^2\int_0^1\left(\int_0^x p_x(x,y,\hat{D}(t))z(y,t)\mathrm{d}y\right)^2\mathrm{d}x,\\
&\le 5\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\|\check{w}\|^2 + 5\bar{D}^2\left(l_y(1,1)^2\right.\\
&\left.+\int_0^1\int_0^x \eta_{xy}(x-y,1,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\right)\|z\|^2 + 5\|z_x\|^2\\
&+\frac{5}{3}z(0,t)^2\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x, \qquad \text{(B.5)}
\end{aligned}
$$

again using Cauchy–Schwarz and Young's inequalities combined with the fact that $p(x,x,\hat{D}(t)) = -\eta_y(0,1,\hat{D}(t)) = -l_y(1,1)$ and $p_x(x,y,\hat{D}(t)) = -\eta_{xy}(x-y,1,\hat{D}(t))$. Since

$$
\begin{aligned}
\int_0^1\int_0^x \eta_{xy}(x-y,1,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x &= \int_0^1\int_0^x \eta_{xy}(s,1,\hat{D}(t))^2\mathrm{d}s\mathrm{d}x,\\
&\le \int_0^1 \eta_{xy}(x,1,\hat{D}(t))^2\mathrm{d}x, \qquad \text{(B.6)}
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
\int_0^1 v_x(x,t)^2\mathrm{d}x \le{}& 5\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x\|\check{w}\|^2\\
&+5\bar{D}^2\left(l_y(1,1)^2+\int_0^1 \eta_{xy}(x,1,\hat{D}(t))^2\mathrm{d}x\right)\|z\|^2\\
&+5\|z_x\|^2+\frac{5}{3}z(0,t)^2\int_0^1\int_0^1 \eta_x(x,y,\hat{D}(t))^2\mathrm{d}y\mathrm{d}x. \qquad \text{(B.7)}
\end{aligned}
$$

From (42) and (54), we have

$$
\begin{aligned}
v(0,t)^2 ={}& \left(z(0,t)+\int_0^1 \eta(0,y,\hat{D}(t))(\check{w}(y,t)+yz(0,t))\mathrm{d}y\right)^2,\\
&\le 3z(0,t)^2+3\left(\int_0^1 \eta(0,y,\hat{D}(t))\check{w}(y,t)\mathrm{d}y\right)^2\\
&+3\left(\int_0^1 \eta(0,y,\hat{D}(t))yz(0,t)\mathrm{d}y\right)^2,\\
&\le 3\int_0^1 l(1,y)^2\mathrm{d}y\|\check{w}\|^2 + z(0,t)^2\left(3+\int_0^1 l(1,y)^2\mathrm{d}y\right), \qquad \text{(B.8)}
\end{aligned}
$$

where $\eta(0,y,\hat{D}(t)) = l(1,y)$. Combining (B.1), (B.4), (B.7), (B.8), Lemmas 1 and 2, one derives (68). Finally, using Lemmas 3 and 4, one can establish (69) in a similar way.

Appendix C. Proof of the boundedness of the M_i functions

The boundedness of the functions M_i, $i = \{1,2,3,4\}$ is established based on Lemmas 2–4 and the following lemmas.

Lemma 5. *For the function $M_1(x,\hat{D}(t))$ defined in (50), the following inequalities hold:*

$$
|M_1(0,\hat{D}(t))| \le |k_y(1,1)|, \qquad \text{(C.1)}
$$

$$
\begin{aligned}
M_1(1,\hat{D}(t))^2 \le{}& k_y(1,1)^2+\int_0^1 \gamma_y(x,1,\hat{D}(t))^2\mathrm{d}x\\
&+\int_0^1 \gamma_{yx}(x,1,\hat{D}(t))^2\mathrm{d}x, \qquad \text{(C.2)}
\end{aligned}
$$

$$
\int_0^1 M_1(x,\hat{D}(t))^2\mathrm{d}x \le \int_0^1 \gamma_y(x,1,\hat{D}(t))^2\mathrm{d}x, \qquad \text{(C.3)}
$$

$$
\int_0^1 M_{1x}(x,\hat{D}(t))^2\mathrm{d}x \le \int_0^1 \gamma_{yx}(x,1,\hat{D}(t))^2\mathrm{d}x. \qquad \text{(C.4)}
$$

Proof. From the function $M_1(x,\hat{D}(t))$ defined in (50), we can get (C.1), (C.3), (C.4) directly. To prove (C.2), we use Agmon's and Young's inequalities to get the following estimates

$$
\begin{aligned}
M_1(1,\hat{D}(t))^2 ={}& \gamma_y(1,1,\hat{D}(t))^2,\\
&\le \gamma_y(0,1,\hat{D}(t))^2+2\left(\int_0^1 \gamma_y(x,1,\hat{D}(t))^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\cdot\left(\int_0^1 \gamma_{yx}(x,1,\hat{D}(t))^2\mathrm{d}x\right)^{\frac{1}{2}},\\
&\le \gamma_y(0,1,\hat{D}(t))^2+\int_0^1 \gamma_y(x,1,\hat{D}(t))^2\mathrm{d}x\\
&+\int_0^1 \gamma_{yx}(x,1,\hat{D}(t))^2\mathrm{d}x. \qquad \text{(C.5)}
\end{aligned}
$$

Since $\gamma(0, y, \hat{D}(t)) = k(1, y)$, we get

$$\gamma_y(0, 1, \hat{D}(t)) = k_y(1, 1), \tag{C.6}$$

and substituting (C.6) into (C.5), one can obtain (C.2) and complete the proof of the lemma. ■

Likewise, we can proof the following three lemmas.

Lemma 6. *For the function $M_2(x, y, \hat{D}(t))$ defined in (51), the following inequalities hold:*

$$\begin{aligned}
&\int_0^1 M_2(0, y, \hat{D}(t))^2 dy \\
&\quad \le 3k_y(1, 1)^2 \int_0^1 l(1, y)^2 dy + 6\int_0^1 (k_{yy}(1, y)^2 + \lambda^2 k(1, y)^2) dy \\
&\quad \cdot \left(1 + \int_0^1 \int_0^x l(x, y)^2 dy dx\right),
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
&\int_0^1 M_2(1, y, \hat{D}(t))^2 dy \le 3M_1(1, \hat{D}(t))^2 \int_0^1 l(1, y)^2 dy \\
&\quad + \frac{3}{\underline{D}^2}\left(2\bar{D}^2 \int_0^1 (k_{yy}(1, y)^2 + \lambda^2 k(1, y)^2) dy\right. \\
&\quad \left. + \int_0^1 \int_0^1 (\gamma_x(x, y, \hat{D}(t))^2 + \gamma_{xx}(x, y, \hat{D}(t))^2) dy dx\right) \\
&\quad \cdot \left(1 + \int_0^1 \int_0^x l(x, y)^2 dy dx\right),
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 M_2(x, y, \hat{D}(t))^2 dy dx \\
&\le \frac{3}{\underline{D}^2} \int_0^1 \int_0^1 \gamma_x(x, y, \hat{D}(t))^2 dy dx \left(1 + \int_0^1 \int_0^x l(x, y)^2 dy dx\right) \\
&\quad + 3\int_0^1 M_1(x, \hat{D}(t))^2 dx \int_0^1 l(1, y)^2 dy,
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 |M_{2x}(x, y, \hat{D}(t))|^2 dy dx \\
&\le \frac{3}{\underline{D}^2} \int_0^1 \int_0^1 \gamma_{xx}(x, y, \hat{D}(t))^2 dy dx \left(1 + \int_0^1 \int_0^x l(x, y)^2 dy dx\right) \\
&\quad + 3\int_0^1 M_{1x}(x, \hat{D}(t))^2 dx \int_0^1 l(1, y)^2 dy.
\end{aligned} \tag{C.10}$$

Lemma 7. *For the function $M_3(x, y, \hat{D}(t))$ defined in (52), the following inequalities hold:*

$$\int_0^1 M_3(x, x, \hat{D}(t))^2 dx \le k_y(1, 1)^2, \tag{C.11}$$

$$\begin{aligned}
&\int_0^1 M_3(1, y, \hat{D}(t))^2 dy \\
&\le 4\int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx \\
&\quad + 4\bar{D}^2 \int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx \\
&\quad \cdot \int_0^1 \eta_y(x, 1, \hat{D}(t))^2 dx,
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
&\int_0^1 \int_0^x M_3(x, y, \hat{D}(t))^2 dy dx \\
&\le 4\int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx \\
&\quad + 4\bar{D}^2 \int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx \\
&\quad \cdot \int_0^1 \eta_y(x, 1, \hat{D}(t))^2 dx,
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
&\int_0^1 \int_0^x M_{3x}(x, y, \hat{D}(t))^2 dy dx \\
&\le 5\int_0^1 (4\gamma_{yx}(x, 1, \hat{D}(t))^2 + \gamma_{yxx}(x, 1, \hat{D}(t))^2) dx \\
&\quad + 5\bar{D}^2 \left(k_y(1, 1)^2 + 4\int_0^1 \gamma_{yx}(x, 1, \hat{D}(t))^2 dx\right. \\
&\quad \left. + \int_0^1 \gamma_{yxx}(x, 1, \hat{D}(t))^2 dx\right) \int_0^1 \eta_y(x, 1, \hat{D}(t))^2 dx.
\end{aligned} \tag{C.14}$$

Lemma 8. *For the function $M_4(x, y, \hat{D}(t))$ defined in (53), $M_4(0, y, \hat{D}(t)) = 0$, and the following inequalities hold:*

$$\begin{aligned}
&\int_0^1 M_4(1, y, \hat{D}(t))^2 dy \\
&\le \frac{4}{\underline{D}^2}\left(2\bar{D}^2 \int_0^1 (k_{yy}(1, y)^2 + \lambda^2 k(1, y)^2) dx\right. \\
&\quad \left. + \int_0^1 \int_0^1 (\gamma_x(x, y, \hat{D}(t))^2 + \gamma_{xx}(x, y, \hat{D}(t))^2) dx\right) \\
&\quad \cdot \left(1 + \int_0^1 \int_0^x l(x, y) dy dx\right) + 4\int_0^1 \int_0^1 \eta(x, y, \hat{D}(t)) dy dx \\
&\quad \cdot \int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx,
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 M_4(x, y, \hat{D}(t))^2 dy dx \\
&\le \frac{4}{\underline{D}^2} \int_0^1 \int_0^1 \gamma_x(x, y, \hat{D}(t))^2 dy dx \left(1 + \int_0^1 \int_0^x l(x, y) dy dx\right) \\
&\quad + 4\int_0^1 (\gamma_y(x, 1, \hat{D}(t))^2 + \gamma_{yx}(x, 1, \hat{D}(t))^2) dx \\
&\quad \cdot \int_0^1 \int_0^1 \eta(x, y, \hat{D}(t)) dy dx,
\end{aligned} \tag{C.16}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 M_{4x}(x, y, \hat{D}(t))^2 dy dx \\
&\le 7\left(\int_0^1 (4\gamma_{yx}(x, 1, \hat{D}(t))^2 + \gamma_{yxx}(x, 1, \hat{D}(t))^2) dx\right. \\
&\quad \left. + k_y(1, 1)^2\right) \int_0^1 \int_0^1 \eta(x, y, \hat{D}(t)) dy dx \\
&\quad + \frac{7}{\underline{D}^2} \int_0^1 \int_0^1 (\gamma_x(x, y, \hat{D}(t))^2 + \gamma_{xx}(x, y, \hat{D}(t))^2) dy dx \\
&\quad \cdot \left(1 + \int_0^1 \int_0^x l(x, y) dy dx\right).
\end{aligned} \tag{C.17}$$

References

Anfinsen, H., & Aamo, O. M. (2017a). Adaptive disturbance rejection in 2 × 2 linear hyperbolic PDEs. In *2017 IEEE 56th annual conference on decision and control (CDC)* (pp. 286–292).

Anfinsen, H., & Aamo, O. M. (2017b). Adaptive output-feedback stabilization of linear 2 × 2 hyperbolic systems using anti-collocated sensing and control. *Systems & Control Letters*, *104*, 86–94.

Anfinsen, H., & Aamo, O. M. (2017c). Adaptive stabilization of $n+1$ coupled linear hyperbolic systems with uncertain boundary parameters using boundary sensing. *Systems & Control Letters*, *99*, 72–84.

Anfinsen, H., & Aamo, O. M. (2018a). Adaptive control of linear 2 × 2 hyperbolic systems. *Automatica*, *87*, 69–82.

Anfinsen, H., & Aamo, O. M. (2018b). A note on establishing convergence in adaptive systems. *Automatica*, *93*, 545–549.
Anfinsen, H., Diagne, M., Aamo, O. M., & Krstic, M. (2016). An adaptive observer design for $n + 1$ coupled linear hyperbolic PDEs based on swapping. *IEEE Transactions on Automatic Control*, *61*(12), 3979–3990.
Anfinsen, H., Diagne, M., Aamo, O. M., & Krstic, M. (2017). Estimation of boundary parameters in general heterodirectional linear hyperbolic systems. *Automatica*, (79), 185–197.
Baccoli, A., Orlov, Y., & Pisano, A. (2014). On the boundary control of coupled reaction–diffusion equations having the same diffusivity parameters. In *53rd IEEE conference on decision and control* (pp. 5222–5228).
Belkoura, L., & Orlov, Y. (2002). Identifiability analysis of linear delay-differential systems. *IMA Journal of Mathematical Control and Information*, *19*, 73–81.
Bentsman, J., & Orlov, Yu. (2001). Reduced spatial order model reference adaptive control of spatially varying distributed parameter systems of parabolic and hyperbolic types. *International Journal of Adaptive Control & Signal Processing*, *15*(6), 679–696.
Bernard, P., & Krstic, M. (2014). Adaptive output-feedback stabilization of non-local hyperbolic PDEs. *Automatica*, *50*(10), 2692–2699.
Boskovic, D. M., & Krstic, M. (2002). Backstepping control of chemical tubular reactors. *Computers & Chemical Engineering*, *26*(7), 1077–1085.
Boskovic, D. M., Krstic, M., & Liu, W. (2001). Boundary control of an unstable heat equation via measurement of domain-averaged temperature. *IEEE Transactions on Automatic Control*, *46*(12), 2022–2028.
Bresch-Pietri, D., & Krstic, M. (2010). Delay-adaptive predictor feedback for systems with unknown long actuator delay. *IEEE Transactions on Automatic Control*, *55*(9), 2106–2112.
Demetriou, M. A., & Rosen, I. G. (1994). On the persistence of excitation in the adaptive estimation of distributed parameter systems. *IEEE Transactions on Automatic Control*, *39*(5), 1117–1123.
Diagne, M., Bekiaris-Liberis, N., & Krstic, M. (2017). Compensation of input delay that depends on delayed input. *Automatica*, *85*, 362–373.
Diagne, M., Bekiaris-Liberis, N., Otto, A., & Krstic, M. (2017). PDE/nonlinear ODE cascades with state-dependent propagation speed. *IEEE Transactions on Automatic Control*, *62*(12), 6278–6293.
Dubljevic, S., Kobilarov, M., & Ng, J. (2010). Discrete mechanics optimal control (DMOC) and model predictive control (MPC) synthesis for reaction–diffusion process system with moving actuator. In *2010 American control conference* (pp. 5694–5701).
Eleiwi, F., & Laleg-Kirati, T. M. (2018). Observer-based perturbation extremum seeking control with input constraints for direct-contact membrane distillation process. *International Journal of Control*, *91*(6), 1363–1375.
Ferrari-Trecate, G., Buffa, A., & Gati, M. (2006). Analysis of coordination in multi-agent systems through partial difference equations. *IEEE Transactions on Automatic Control*, *51*(6), 1058–1063.
Forman, J. C., Bashash, S., Stein, J., & Fathy, H. (2011). Reduction of an electrochemistry-based li-ion battery health degradation model via constraint linearization and padé approximation. In *ASME 2010 dynamic systems and control conference, Vol. 2* (pp. 173–183).
Guzmán, P., Marx, S., & Cerpa, E. (2019). Stabilization of the linear Kuramoto-Sivashinsky equation with a delayed boundary control. *IFAC-PapersOnLine*, *52*(2), 70–75.
Koga, S., Diagne, M., & Krstic, M. (2019). Control and state estimation of the one-phase Stefan problem via backstepping design. *IEEE Transactions on Automatic Control*, *64*(2), 510–525.
Krstic, M. (2009a). Compensating actuator and sensor dynamics governed by diffusion PDEs. *Systems & Control Letters*, *58*(5), 372–377.
Krstic, M. (2009b). Control of an unstable reaction–diffusion PDE with long input delay. *Systems & Control Letters*, *58*(10), 773–782.
Krstic, M., & Bresch-Pietri, D. (2009). Delay-adaptive full-state predictor feedback for systems with unknown long actuator delay. In *2009 American control conference* (pp. 4500–4505).
Krstic, M., Kanellakopoulos, I., & Kokotovic, P. (1995). *Nonlinear and adaptive control design*. New York: Wiley.
Krstic, M., & Smyshlyaev, A. (2008). Adaptive boundary control for unstable parabolic PDEs-part I: Lyapunov design. *IEEE Transactions on Automatic Control*, *53*(7), 1575–1591.
Lei, C., Lin, Z., & Wang, H. (2013). The free boundary problem describing information diffusion in online social networks. *Journal of Differential Equations*, *254*(3), 1326–1341.
Lhachemi, H., & Prieur, C. (2021). Feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control. *IEEE Transactions on Automatic Control*, *66*(1), 105–120.
Lhachemi, H., Prieur, C., & Shorten, R. (2019). An LMI condition for the robustness of constant-delay linear predictor feedback with respect to uncertain time-varying input delays. *Automatica*, *109*, Article 108551.
Meurer, T., Becker, J., & Zeitz, M. (2003). Flatness-based feedback tracking control of a distributed parameter tubular reactor model. In *2003 European control conference (ECC)* (pp. 2893–2898).
Meurer, T., & Kugi, A. (2009). Trajectory planning for boundary controlled parabolic PDEs with varying parameters on higher-dimensional spatial domains. *IEEE Transactions on Automatic Control*, *54*(8), 1854–1868.
Meurer, T., & Zeitz, M. (2003). A novel design approach to flatness-based feedback boundary control of nonlinear reaction–diffusion systems with distributed parameters. In *New trends in nonlinear dynamics and control and their applications* (pp. 221–235).
Orlov, Y., & Dochain, D. (2002). Discontinuous feedback stabilization of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor. *IEEE Transactions on Automatic Control*, *47*(8), 1293–1304.
Orlov, Y., Fradkov, A., & Andrievsky, B. (2020). Output feedback energy control of the sine-Gordon PDE model using collocated spatially sampled sensing and actuation. *IEEE Transactions on Automatic Control*, *65*(4), 1484–1498.
Prieur, C., & Trelat, E. (2019). Feedback stabilization of a 1-D linear reaction–diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, *64*(4), 1415–1425.
Qi, J., & Krstic, M. (2021). Compensation of spatially-varying input delay in distributed control of reaction–diffusion PDEs. *IEEE Transactions on Automatic Control*, http://dx.doi.org/10.1109/TAC.2020.3027662.
Qi, J., Tang, S., & Wang, C. (2019). Parabolic PDE-based multi-agent formation control on a cylindrical surface. *International Journal of Control*, *92*(1), 77–99.
Qi, J., Vazquez, R., & Krstic, M. (2015). Multi-agent deployment in 3-D via PDE control. *IEEE Transactions on Automatic Control*, *60*(4), 891–906.
Qi, J., Wang, S., Fang, J., & Diagne, M. (2019). Control of multi-agent systems with input delay via PDE-based method. *Automatica*, *106*, 91–100.
Smyshlyaev, A., & Krstic, M. (2010). *Daptive control of parabolic PDEs*. Princeton University Press.
Smyshlyaev, A., Orlov, Y., & Krstic, M. (2010). Adaptive identification of two unstable PDEs with boundary sensing and actuation. *International Journal of Adaptive Control and Signal Processing*, *23*(2), 131–149.
Tang, S., & Xie, C. (2011). State and output feedback boundary control for a coupled PDE-ODE system. *Systems & Control Letters*, *60*(8), 540–545.
Vazquez, R., & Krstic, M. (2017). Boundary control of coupled reaction-advection-diffusion systems with spatially-varying coefficients. *IEEE Transactions on Automatic Control*, *62*(4), 2026–2033.
Wang, S., Qi, J., & Fang, J. (2017). Control of 2-D reaction-advection-diffusion PDE with input delay. In *2017 Chinese automation congress (CAC)* (pp. 7145–7150).
Zhu, Y., & Krstic, M. (2015). Adaptive backstepping control of uncertain linear systems under unknown actuator delay. *Automatica*, *59*, 256–265.
Zhu, Y., Krstic, M., & Su, H. (2017). Adaptive output feedback control for uncertain linear time-delay systems. *IEEE Transactions on Automatic Control*, *62*(2), 545–560.

Shanshan Wang received the B.S. degree in Electrical Engineering and Automation from Nanhang Jincheng College, NanJing, China, in 2015. She is currently a Ph.D student with the Department of Control Science and Engineering of the Donghua University, Shanghai, China. She was a visiting student with the Department of Mechanical Aerospace and Nuclear Engineering at Rensselaer Polytechnic Institute, Troy, New York, from November 2018 to December 2019. Her research interests include multi-agent cooperative control, the control of delay systems and adaptive control.

Jie Qi is a Professor of the Automation Department, Donghua University, China. She received her Ph.D. degree in Systems Engineering (2005) and the B.S. degree in Automation (2000) from Northeastern University in Shenyang, China. She was a research fellow with the Institute of Textiles & Clothing, The Hong Kong Polytechnic University, Hong Kong from 2007 to 2008; a visiting researcher with the Cymer Center for Control Systems and Dynamics at the University of California, San Diego, from March 2013 to February 2014 and from June to September in 2015; and a visiting researcher with the Chemical and Materials Engineering Department at the University of Alberta, from January 2019 to January 2020. Her research interests include control and estimation of distributed parameters systems, control of delayed systems and its applications on multi-agent systems and industry process.

Mamadou Diagne is currently an Assistant Professor with the Department of Mechanical Aerospace and Nuclear Engineering at Rensselaer Polytechnic Institute, Troy, New York. He received his Ph.D. degree in 2013 at Laboratoire d'Automatique et du Génie des Procédés, Université Claude Bernard Lyon I (France). He was first a postdoctoral fellow at the Cymer Center for Control Systems and Dynamics of UC San Diego from 2013 to 2015 and then at the Department of Mechanical Engineering of the University of Michigan from 2015 to 2016. His research interests concern the modeling of flow systems involving heat and mass transport phenomena and the control of PDEs, mixed PDE/ODEs, and delay systems. He received the NSF CAREER in 2020.